

EXPONENTIAL MIXING OF TORUS EXTENSIONS OVER EXPANDING MAPS

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ABSTRACT. We study the mixing property for the skew product $F : \mathbb{T}^d \times \mathbb{T}^\ell \rightarrow \mathbb{T}^d \times \mathbb{T}^\ell$ given by $F(x, y) = (Tx, y + \tau(x) \pmod{\mathbb{Z}^\ell})$, where $T : \mathbb{T}^d \rightarrow \mathbb{T}^d$ is a C^∞ uniformly expanding endomorphism, and the fiber map $\tau : \mathbb{T}^d \rightarrow \mathbb{R}^\ell$ is a C^∞ map. We apply the semiclassical approach to get the dichotomy: either F mixes exponentially fast or τ is an essential coboundary. In the former case, the Koopman operator \hat{F} of F has spectral gap in some Hilbert space \mathcal{W}^s , $s < 0$, which contains all $(-s)$ -Hölder continuous functions on $\mathbb{T}^d \times \mathbb{T}^\ell$; and in the latter case, either F is not weak mixing, or it can be approximated by non-mixing skew products that are semiconjugate to circle rotations.

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0. INTRODUCTION.

In this paper we study the mixing properties for torus extension of expanding maps. The systems F we consider are of the form of skew products with expanding $T : \mathbb{T}^d \rightarrow \mathbb{T}^d$ on the base and torus rotations with rotation vectors $\tau(x)$, $x \in \mathbb{T}^d$, on the fibers \mathbb{T}^ℓ . (See (1.2) for the maps.) We obtain a dichotomy: either such a

system has exponential decay of correlations with respect to the smooth invariant measure, or the rotation function $\tau(x)$ over \mathbb{T}^d is an essential coboundary over T . When the base map T is fixed, the former case is open and dense in the C^0 topology of the space of skew products. The latter implies that either the system is not weak mixing and is semiconjugate to an expanding endomorphism crossing a circle rotation, or it is unstably mixing and can be approximated by non-mixing skew products.

The method we use to get exponential mixing is the semiclassical analysis approach. Instead of the Ruelle-Perron-Frobenius transfer operators acting on some Hölder function space, we study the dual operator, Koopman operator \hat{F} , given by $\hat{F}\phi = \phi \circ F$, acting on certain distribution space. By Fourier transform along \mathbb{T}^ℓ , the fiber direction, the operator can be decomposed to a family of operators $\{\hat{F}_\nu\}_{\nu \in \mathbb{Z}^\ell}$, where ν is the frequency. Such operators can be regarded as Fourier integral operators. Using semiclassical analysis theory we show that if τ is not an essential coboundary, then the spectral radius of \hat{F}_ν is strictly less than 1 for all $\nu \neq \mathbf{0}$, while 1 is the only leading eigenvalue of $\hat{F}_\mathbf{0}$ on the unit circle and it is simple. Further, we obtain uniform operator norm control on the iterates $\{\hat{F}_\nu^n\}_{n \in \mathbb{N}}$ for all ν of sufficiently large magnitude. We hence prove that the Koopman operator \hat{F} has a spectral gap, and the system has exponential decay of correlations.

By analyzing the associated RPF transfer operators, Dolgopyat established in [9] the exponential mixing property for compact Lie group extensions of expanding maps under a generic condition called infinitesimally completely non-integrability, which is equivalent to the non-coboundary condition of $\tau(x)$ if the group is a torus. The crucial technique used there is now called Dolgopyat's oscillatory cancellation argument, and it has been successfully developed to study the rate of mixing for various systems with neutral direction, see [8], [18], [3], [4], etc. Among all such results is [7], in which Butterley and Eslami obtained a dichotomy similar to that in our main theorem (Theorem 1) for a piecewise C^2 circle extension of a circle expanding map. We remark that their analysis did not provide finer structures of the dynamics, such as the spectral properties of the transfer operators, at least not in an explicit form.

In a somewhat different direction, the semiclassical analysis approach is used to study Ruelle-Pollicott resonances for some hyperbolic systems, see [11], [13], [14], etc. Applying this approach in the context of partially hyperbolic systems, Faure showed in [12] that a simple but intuitive model – a circle extension of a circle expanding map – has exponential decay of correlations under a so-called partially captive condition. It was recently pointed out in [20] that the partially captive condition is generic and equivalent to the non-coboundary condition of $\tau(x)$ for this two-dimensional model. Using similar techniques, Arnoldi established in [1] the asymptotic spectral gap and the fractal Weyl law for $SU(2)$ extensions of circle expanding maps under the partially captive condition, and later Arnoldi, Faure and Weich obtained a similar result in [2] for circle extensions of certain one-dimensional open expanding maps under a stronger condition called minimal captivity.

Let us mention some similar results in the context of suspension semi-flows over expanding maps. Pollicott [23] showed that a generic suspension semi-flow over an expanding Markov interval map is exponentially mixing. In the case when the base is a linear expanding map, Tsujii [28] constructed an anisotropic Sobolev space on

which the transfer operator has spectral gap. Also, Baladi and Vallée [5] proved exponential mixing property for surface semi-flows without finite Markov partitions.

The main technique we use in this paper is the semiclassical analysis method, inspired by Faure [12] and other related works. A key ingredient in our analysis is that we introduce non-standard Sobolev spaces associated with dynamical weights. Although equivalent to the standard ones, these spaces are much more effective in extracting the spectral properties of the Koopman operator \hat{F} and its decompositions $\{\hat{F}_\nu\}_{\nu \in \mathbb{Z}^\ell}$. In fact, we convert each \hat{F}_ν^n by unitary conjugation into a pseudo-differential operator, whose symbol provides an upper bound for the operator norm of \hat{F}_ν^n . Moreover, we prove directly that the rotation vector $\tau(x)$ is not an essential coboundary if and only if those upper bounds vanish uniformly exponentially fast as $n \rightarrow \infty$ for all high frequencies ν , from which we conclude that \hat{F} has spectral gap. We remark that our approach bypasses the captive conditions, and has no dimension restrictions on either the base or the fiber.

This paper is organized as the following. The setting and statements of results are given in Section 1. In Section 2 we introduce some notions and results from classical and semiclassical analysis, including Fourier transform, Sobolev spaces, Pseudo-differential operators, Fourier Integral Operators, Egorov's Theorem, and L^2 -continuity theorems. This section is not necessary for the reader who is familiar with the theory. We prove the theorems of the paper in Section 3 based on Proposition 3.1 and 3.2, which give the spectral radius of the Koopman operator, the dual operator of the transfer operator. The propositions are proved in Section 4, using classical and semiclassical analysis. A key estimates in the proof, stated in Lemma 5.1, is postponed in Section 5.

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1. STATEMENT OF RESULTS.

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let $T : \mathbb{T}^d \rightarrow \mathbb{T}^d$ be a C^∞ uniformly expanding map such that

$$(1.1) \quad \gamma := \inf_{(x,v) \in S\mathbb{T}^d} |D_x T(v)| > 1,$$

where $S\mathbb{T}^d$ is the unit tangent bundle over \mathbb{T}^d . It is well known that T has a unique smooth invariant probability measure $d\mu(x) = h(x)dx$, where the density function $h \in C^\infty(\mathbb{T}^d, \mathbb{R}^+)$. Further, T is mixing with respect to μ . Here and throughout this paper, we fix the expanding map T .

Given a function $\tau \in C^\infty(\mathbb{T}^d, \mathbb{R}^\ell)$, we define the skew product $F = F_\tau : \mathbb{T}^d \times \mathbb{T}^\ell \rightarrow \mathbb{T}^d \times \mathbb{T}^\ell$ by

$$(1.2) \quad F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Tx \\ y + \tau(x) \pmod{\mathbb{Z}^\ell} \end{pmatrix},$$

which preserves the product measure $dA = d\mu(x)dy$.

The mixing property of the system $(\mathbb{T}^{d+\ell}, F, dA)$ is quantified by the rates of decay of correlations. We say that the skew product F is *exponentially mixing* with respect to the smooth measure dA if there exists $\rho \in [0, 1)$ such that for any pair

of Hölder observables $\phi, \psi \in C^\alpha(\mathbb{T}^{d+\ell})$, $\alpha > 0$, the correlation function

$$C_n(\phi, \psi; F, dA) = \left| \int \phi \circ F^n \cdot \psi dA - \int \phi dA \int \psi dA \right|$$

satisfies $C_n(\phi, \psi; F, dA) \leq C_{\phi, \psi} \rho^n$ for all $n \geq 1$, where $C_{\phi, \psi} > 0$ is a constant depending on ϕ and ψ .

Certain cohomological conditions might give obstructions to the exponential mixing property.

Definition 1.1. A real-valued function $\varphi \in C^\infty(\mathbb{T}^d, \mathbb{R})$ is called an essential coboundary over T if there exist $c \in \mathbb{R}$ and a measurable function $u : \mathbb{T}^d \rightarrow \mathbb{R}$ such that

$$\varphi(x) = c + u(x) - u(Tx), \quad \mu - \text{a.e. } x.$$

Let \mathfrak{B} be the space of real-valued essential coboundaries over T .

A vector-valued function $\tau = (\tau_1, \tau_2, \dots, \tau_\ell) \in C^\infty(\mathbb{T}^d, \mathbb{R}^\ell)$ is called an essential coboundary over T if $\tau_1, \tau_2, \dots, \tau_\ell$ are linearly dependent mod \mathfrak{B} , that is, there exist $v \in \mathbb{R}^\ell \setminus \{\mathbf{0}\}$, $c \in \mathbb{R}$ and a measurable function $u : \mathbb{T}^d \rightarrow \mathbb{R}$ such that

$$(1.3) \quad v \cdot \tau(x) = c + u(x) - u(Tx), \quad \mu - \text{a.e. } x. \quad ^1$$

Remark 1.2.

(i) By Livsic theory (see e.g. [17]), the measurable function $u : \mathbb{T}^d \rightarrow \mathbb{R}$ in (1.3) is in fact of class C^∞ .

(ii) The functions $\tau_1, \tau_2, \dots, \tau_\ell$ are called integrally dependent mod \mathfrak{B} if (1.3) holds for some $v \in \mathbb{Z}^\ell \setminus \{\mathbf{0}\}$. There are functions $\tau_1, \tau_2, \dots, \tau_\ell$ that are linearly dependent but not integrally dependent mod \mathfrak{B} , unless $\ell = 1$.

Our main result is the following.

Theorem 1. Let $(\mathbb{T}^{d+\ell}, F, dA)$ be the skew product as described above. We have the following dichotomy:

- (1) Either F is exponentially mixing (with respect to dA);
- (2) Or $\tau(x)$ is an essential coboundary over T .

If $d = \ell = 1$, the above dichotomy is proved by Butterley and Eslami [7], in which the circle expansion T and the rotation τ are allowed to have a finite number of discontinuities.

Remark 1.3. The second case in Theorem 1 is very rare in the sense that the closed subspace that consists of all essential coboundaries has infinite codimension in $C^\infty(\mathbb{T}^d, \mathbb{R}^\ell)$. It means that the first case is generic in the space of skew products, that is, there is an open and dense subset \mathcal{U} in $C^\infty(\mathbb{T}^d, \mathbb{R}^\ell)$ under the C^0 topology such that for all $\tau \in \mathcal{U}$, the corresponding skew product F_τ is exponentially mixing.

The infinite codimension of essential coboundaries is a crucial property in showing the stable ergodicity of skew products over general hyperbolic systems. Among tremendous results on this topic, we refer the reader to [21], [16], [6], [15], etc.

¹ Here “ \cdot ” denotes the standard inner product of two vectors in \mathbb{R}^ℓ . In the rest of the paper, we shall abuse the notation $v \cdot w$ when one of v and w belongs to \mathbb{Z}^ℓ , \mathbb{T}^ℓ or $\mathbb{S}^{\ell-1}$ - the unit sphere in \mathbb{R}^ℓ ; that is, $v \cdot w$ represents the inner product of v and w as vectors in \mathbb{R}^ℓ . The vector representation of $v \in \mathbb{Z}^\ell$ or $\mathbb{S}^{\ell-1}$ is obvious; for $v \in \mathbb{T}^\ell$, we naturally choose the representative vector in $[0, 1)^\ell \subset \mathbb{R}^\ell$.

In our context, we say that $F = F_\tau$ is stably ergodic if $F_{\tau'}$ is ergodic for any τ' that is C^0 -close to τ . The stable mixing property and stable exponential mixing property are defined in a similar fashion.

It was shown by Parry and Pollicott [21], and also by Field and Parry [16], that $(\mathbb{T}^{d+\ell}, F, dA)$ is weak mixing and stably mixing if and only if the functions $\tau_1, \tau_2, \dots, \tau_\ell$ are integrally independent and linearly independent mod \mathfrak{B} , respectively.² In other words, Theorem 1 asserts that if F is stably mixing, then it is exponentially mixing, and furthermore, it is stably exponentially mixing by Remark 1.3. We can say more about the ergodic properties of the skew product over an expanding map: Dolgopyat [9] proved that F is stably ergodic if and only if it is exponentially mixing; Field and Parry [16] showed that stable ergodicity implies stable mixing property for skew products. Combining all these results and Theorem 1, we immediately obtain the following corollary.

Corollary 1.4. *Let F be the skew product given by (1.2). The following statements are equivalent:*

- (1) F is stably ergodic;
- (2) F is stably mixing;
- (3) F is exponentially mixing;
- (4) F is stably exponentially mixing.

Remark 1.5.

(i) In the case when $\ell = 1$, if F is mixing, then F is stably mixing and thus (stably) exponentially mixing. This is simply because that integral independence and linear independence mod \mathfrak{B} are the same for $\tau \in C^\infty(\mathbb{T}^d, \mathbb{R})$.

(ii) We thank one of the referees for providing us the following unstably mixing example. Let $F_0 : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be given by

$$F_0(x, y_1, y_2) = (2x, y_1 + \tau_0(x), y_2 + \sqrt{3}\tau_0(x)) \pmod{\mathbb{Z}^3},$$

where $\tau_0(x)$ is not a real-valued essential coboundary over the linear expanding map $x \mapsto 2x \pmod{\mathbb{Z}}$ of the circle. It is clear that $\tau_0(x)$ and $\sqrt{3}\tau_0(x)$ are integrally independent but not linearly independent mod \mathfrak{B} , and hence F_0 is unstably mixing and thus unstably ergodic by [21], [16].

We shall follow the semiclassical analysis approach in [12] to prove Theorem 1. Instead of the Ruelle-Perron-Frobenius transfer operators acting on some Hölder function space, we study the dual operator, Koopman operator, acting on certain distribution space.

More precisely, recall that the Koopman operator $\hat{F} : L^2(\mathbb{T}^{d+\ell}, dA) \rightarrow L^2(\mathbb{T}^{d+\ell}, dA)$ defined by $\hat{F}\phi = \phi \circ F$ is an isometry. Note that dA is equivalent to the Lebesgue measure $dxdy$, we instead study the action of \hat{F} on $L^2(\mathbb{T}^{d+\ell}) := L^2(\mathbb{T}^{d+\ell}, dxdy)$ as well as $\mathcal{D}'(\mathbb{T}^{d+\ell})$, the space of distributions on $\mathbb{T}^{d+\ell}$.

We say that the operator \hat{F} from a Banach space to itself has *spectral gap* if its spectrum

$$(1.4) \quad \text{Spec}(\hat{F}) = \{1\} \cup \mathcal{K},$$

² Originally in [16], the independence is modulo $\mathbf{V} + \mathfrak{B}$ for some finite-dimensional subspace \mathbf{V} of $C^\infty(\mathbb{T}^d, \mathbb{R})$. In our setting, $\mathbf{V} = \{0\}$ because the rotation function τ is null-homotopic in $C^\infty(\mathbb{T}^d, \mathbb{T}^\ell)$ if regarded as a \mathbb{T}^ℓ -valued function.

where 1 is a simple eigenvalue and \mathcal{K} is a compact subset of the unit open disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem 2. *If $\tau(x)$ is not an essential coboundary over T , then there is an \hat{F} -invariant Hilbert subspace $\mathcal{W}^s \subset \mathcal{D}'(\mathbb{T}^{d+\ell})$ such that $\hat{F}|_{\mathcal{W}^s}$ has spectral gap.*

Here $s < 0$ is a negative order, and we shall see that both \mathcal{W}^s and its dual space $(\mathcal{W}^s)'$ contain the Hölder function space $C^{-s}(\mathbb{T}^{d+\ell})$. We will specify the construction of the Hilbert space \mathcal{W}^s in Subsection 2.1.3 (see (2.6)), prove the theorem in Section 3.3, and then show how Theorem 2 implies Theorem 1 in Section 3.4.

Remark 1.6. *It is well known that $\hat{F}|_{L^2(\mathbb{T}^{d+\ell})}$ does not have spectral gap. We get the result of Theorem 2 since the norm of \mathcal{W}^s is weaker along \mathbb{T}^d -direction and stronger along \mathbb{T}^ℓ -direction than that of $L^2(\mathbb{T}^{d+\ell})$.*

Next we characterize the dynamical properties of $F = F_\tau$ when the rotation vector $\tau = (\tau_1, \tau_2, \dots, \tau_\ell)$ is an essential coboundary, that is, the functions $\tau_1, \tau_2, \dots, \tau_\ell$ are linearly dependent mod \mathfrak{B} . There are two cases:

- (1) If $\tau_1, \tau_2, \dots, \tau_\ell$ are integrally dependent mod \mathfrak{B} , then the behaviors of F_τ in the \mathbb{T}^ℓ direction become very simple, as we see in Part (iii) of the next theorem. In particular, F_τ is not weak mixing.
- (2) If $\tau_1, \tau_2, \dots, \tau_\ell$ are linearly dependent but integrally independent mod \mathfrak{B} , then F_τ is unstably mixing. We can approximate F_τ by a sequence of non-mixing skew products $F_{\tau(n)}$ as follows. Pick real-valued sequences $\{c_{n,i}\}_{n \in \mathbb{N}}$, $i = 1, 2, \dots, \ell$, such that $\lim_{n \rightarrow \infty} c_{n,i} = 1$ and $c_{n,1}\tau_1, c_{n,2}\tau_2, \dots, c_{n,\ell}\tau_\ell$ are integrally dependent mod \mathfrak{B} . Then set $\tau(n) = (c_{n,1}\tau_1, c_{n,2}\tau_2, \dots, c_{n,\ell}\tau_\ell)$.

A foliation \mathcal{L} of a smooth manifold M is of dimensional m if the leaves of \mathcal{L} are m dimensional submanifolds. For a smooth dynamical system (F, M) , a foliation \mathcal{L} of M is F invariant if F preserves the leaves, that is, $F(\mathcal{L}(z)) = \mathcal{L}(F(z))$ for any $z \in M$, where $\mathcal{L}(z)$ is the leaf of \mathcal{L} containing z .

A smooth dynamical system (F, M) is semiconjugate to a smooth system (G, N) if there is a smooth map $\pi : M \rightarrow N$ such that $\pi \circ F = G \circ \pi$.

Theorem 3. *Let $F = F_\tau : \mathbb{T}^d \times \mathbb{T}^\ell \rightarrow \mathbb{T}^d \times \mathbb{T}^\ell$ be defined as in (1.2). The following conditions are equivalent.*

- (i) $\tau_1, \tau_2, \dots, \tau_\ell$ are integrally dependent mod \mathfrak{B} ;
- (ii) There is an F invariant $d + \ell - 1$ dimensional foliation \mathcal{L} of $\mathbb{T}^d \times \mathbb{T}^\ell$ and a vector $v \in \mathbb{Z}^\ell \setminus \{0\}$ such that restricted to each fiber $\{x\} \times \mathbb{T}^\ell$, the leaves of $\mathcal{L}|_{\{x\} \times \mathbb{T}^\ell}$ are of $\ell - 1$ dimensional and normal to v .
- (iii) F is semiconjugate to the map $G = T \times R_c : \mathbb{T}^d \times \mathbb{T} \rightarrow \mathbb{T}^d \times \mathbb{T}$ through a continuous map $\pi : \mathbb{T}^d \times \mathbb{T}^\ell \rightarrow \mathbb{T}^d \times \mathbb{T}$, where $R_c : \mathbb{T} \rightarrow \mathbb{T}$ is a circle rotation with rotation number $c \in \mathbb{R}$. Further, F is semiconjugate to R_c .
- (iv) F is not weak mixing.

2. SEMICLASSICAL ANALYSIS: PRELIMINARIES

In this section we introduce some notions and basic properties in semiclassical analysis which we are going to use. The distribution spaces and Sobolev spaces will be used in construction of the Hilbert space \mathcal{W}^s in Theorem 2. The pseudo-differential operators (PDO) and Fourier integral operators (FIO) will be used to prove Proposition 3.1 and 3.2, where the Egorov's theorems and the L^2 -continuity

theorems are also used. For more information and details on the general theory of PDOs and FIOs, one can see in standard references (e.g. [10, 19, 26, 29]). The underlying manifold that we analyze is the torus, on which the PDO and FIO can be globally defined and treated. We shall be more specific on this subject in Section 2.2.2. For the reader who is not familiar with it, we recommend Chapter 4 in [25].

2.1. Function spaces.

2.1.1. *Distribution spaces.* Let $\mathcal{D}(\mathbb{T}^{d+\ell}) = C^\infty(\mathbb{T}^{d+\ell})$. Its dual space $\mathcal{D}'(\mathbb{T}^{d+\ell})$ is the space of distributions on $\mathbb{T}^{d+\ell}$. We denote by $\phi(\psi)$ the action of a distribution $\phi \in \mathcal{D}'(\mathbb{T}^{d+\ell})$ on a function $\psi \in \mathcal{D}(\mathbb{T}^{d+\ell})$. The inclusion $\mathcal{D}(\mathbb{T}^{d+\ell}) \subset \mathcal{D}'(\mathbb{T}^{d+\ell})$ is interpreted by

$$(2.1) \quad \phi(\psi) := \int_{\mathbb{T}^{d+\ell}} \phi(x, y) \psi(x, y) \, dx dy,$$

for any $\phi, \psi \in \mathcal{D}(\mathbb{T}^{d+\ell})$. Since $\hat{F}\mathcal{D}(\mathbb{T}^{d+\ell}) \subset \mathcal{D}(\mathbb{T}^{d+\ell})$, we define the dual operator $\hat{F}' : \mathcal{D}'(\mathbb{T}^{d+\ell}) \rightarrow \mathcal{D}'(\mathbb{T}^{d+\ell})$ by the duality

$$(\hat{F}'\psi)(\phi) = \psi(\hat{F}\phi) \quad \text{for any } \phi \in \mathcal{D}(\mathbb{T}^{d+\ell}), \psi \in \mathcal{D}'(\mathbb{T}^{d+\ell}).$$

It is easy to check that \hat{F}' maps $\mathcal{D}(\mathbb{T}^{d+\ell})$ to itself, and $\hat{F}' : \mathcal{D}(\mathbb{T}^{d+\ell}) \rightarrow \mathcal{D}(\mathbb{T}^{d+\ell})$ is exactly the RPF (Ruelle-Perron-Frobenius) transfer operator over $F : \mathbb{T}^{d+\ell} \rightarrow \mathbb{T}^{d+\ell}$, that is,

$$\hat{F}'\psi(x, y) = \sum_{F(z, w) = (x, y)} \frac{\psi(z, w)}{|\text{Jac}(F)(z, w)|} \quad \text{for any } \psi \in \mathcal{D}(\mathbb{T}^{d+\ell}).$$

Then we extend \hat{F} on $\mathcal{D}'(\mathbb{T}^{d+\ell})$ via the duality again by

$$(\hat{F}\phi)(\psi) = \phi(\hat{F}'\psi) \quad \text{for any } \psi \in \mathcal{D}(\mathbb{T}^{d+\ell}), \phi \in \mathcal{D}'(\mathbb{T}^{d+\ell}).$$

2.1.2. *Sobolev spaces.* The *Fourier transform* of $\varphi \in \mathcal{D}(\mathbb{T}^d)$ is defined by

$$(2.2) \quad \hat{\varphi}(\xi) = \int_{\mathbb{T}^d} \varphi(x) e^{-i2\pi x \cdot \xi} dx, \quad \xi \in \mathbb{Z}^d.$$

The *inverse transform* is given by

$$(2.3) \quad \varphi(x) = \sum_{\xi \in \mathbb{Z}^d} \hat{\varphi}(\xi) e^{i2\pi \xi \cdot x}, \quad x \in \mathbb{T}^d.$$

Denote $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$, and introduce the standard s -inner product

$$(2.4) \quad \langle \varphi, \psi \rangle_s = \sum_{\xi \in \mathbb{Z}^d} \langle \xi \rangle^{2s} \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)}, \quad \varphi, \psi \in \mathcal{D}(\mathbb{T}^d),$$

for any $s \in \mathbb{R}$. The Sobolev space $H^s(\mathbb{T}^d)$ is the completion of $\mathcal{D}(\mathbb{T}^d)$ under $\langle \cdot, \cdot \rangle_s$.

Proposition 2.1. *Sobolev spaces have the following properties:*

- (i) $\mathcal{D}(\mathbb{T}^d) \subset H^s(\mathbb{T}^d) \subset \mathcal{D}'(\mathbb{T}^d)$ for any $s \in \mathbb{R}$;
- (ii) $H^0(\mathbb{T}^d) = L^2(\mathbb{T}^d)$, and $H^s(\mathbb{T}^d) = \{\varphi : D_x^\beta \varphi \in L^2(\mathbb{T}^d) \text{ for any } |\beta| \leq s\}$ if $s \in \mathbb{N}$, where $D_x^\beta \varphi$ are weak derivatives of φ ;
- (iii) $H^s(\mathbb{T}^d) \subset H^{s'}(\mathbb{T}^d)$ if $s > s'$;
- (iv) $C^s(\mathbb{T}^d) \subset H^s(\mathbb{T}^d)$ for all $s \geq 0$, and if $s > \frac{d}{2}$, then $H^s(\mathbb{T}^d) \subset C^{s-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)$ for any small $\varepsilon > 0$;

- (v) the dual space of $H^s(\mathbb{T}^d)$, $s > 0$, is $H^{-s}(\mathbb{T}^d)$, and the dual action of $\phi \in L^2(\mathbb{T}^d) \subset H^{-s}(\mathbb{T}^d)$ on the function $\psi \in H^s(\mathbb{T}^d)$ is given by (2.1).

For technical treatments, besides the standard s -inner product given in (2.4), we will also use t -scaled s -inner product on $H^s(\mathbb{T}^d)$ for $t > 0$, that is,

$$(2.5) \quad \langle \varphi, \psi \rangle_{s,t} = t^{2s} \langle \varphi, \psi \rangle_s = \sum_{\xi \in \mathbb{Z}^d} t^{2s} \langle \xi \rangle^{2s} \widehat{\varphi}(\xi) \overline{\widehat{\psi}(\xi)}.$$

When equipped with $\langle \cdot, \cdot \rangle_{s,t}$, the space is denoted by $H_t^s(\mathbb{T}^d)$. See Section 2.2.7 and 3.1 for our particular choices of the scaling factor t . We shall also introduce another different but equivalent inner product on $H^s(\mathbb{T}^d)$ in Section 4.1.

2.1.3. *The Hilbert space \mathcal{W}^s .* The Hilbert space \mathcal{W}^s that we will use in Theorem 2 is of the form

$$(2.6) \quad \mathcal{W}^s = H^s(\mathbb{T}^d) \otimes H^{-s}(\mathbb{T}^\ell)$$

for some $s < 0$, equipped with the inner product given by

$$\langle \varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2 \rangle_{\mathcal{W}^s} = \langle \varphi_1, \varphi_2 \rangle_{H^s(\mathbb{T}^d)} \langle \psi_1, \psi_2 \rangle_{H^{-s}(\mathbb{T}^\ell)}$$

and extended by linearity.³ We shall give a more explicit formula of $\langle \cdot, \cdot \rangle_{\mathcal{W}^s}$ in Section 3.1.

Remark 2.2. By Proposition 2.1(ii) and (iii), we have $L^2(\mathbb{T}^d) \subset H^s(\mathbb{T}^d)$ and $C^{-s}(\mathbb{T}^\ell) \subset H^{-s}(\mathbb{T}^\ell)$ when $s < 0$, thus

$$\mathcal{W}^s \supset L^2(\mathbb{T}^d) \otimes C^{-s}(\mathbb{T}^\ell) \supset C^{-s}(\mathbb{T}^{d+\ell}).$$

Similarly, the dual space of $(\mathcal{W}^s)' = H^{-s}(\mathbb{T}^d) \otimes H^s(\mathbb{T}^\ell)$ contains $C^{-s}(\mathbb{T}^{d+\ell})$ as well. By Proposition 2.1(i), it is obvious that \mathcal{W}^s can be regarded as a subspace of $\mathcal{D}'(\mathbb{T}^{d+\ell})$.

2.2. **Semiclassical analysis on the torus.** Due to the group structure of \mathbb{T}^d and its dual group \mathbb{Z}^d , a pseudo-differential operator (PDO) on \mathbb{T}^d might be associated with a so-called toroidal symbol defined on $\mathbb{T}^d \times \mathbb{Z}^d$. With differential in the coordinate variable $x \in \mathbb{T}^d$ and difference in the momentum variable $\xi \in \mathbb{Z}^d$, the rules of toroidal symbol calculus is parallel to those of the canonical symbol calculus on the Euclidean space \mathbb{R}^{2d} . It turns out that the toroidal quantization is equivalent to the standard quantization by using Euclidean local charts of the torus. Similarly, a Fourier integral operator (FIO) on \mathbb{T}^d can be dealt by its series representation, which is called the toroidal Fourier series operator (FSO). We refer the reader to Chapter 4 in [25] for details on this subject.

By smooth interpolation in ξ , we can extend a toroidal symbol to an equivalent periodic Euclidean symbol defined on $\mathbb{T}^d \times \mathbb{R}^d \cong T^*\mathbb{T}^d$, whose periodization fits in Hörmander's definition of classical $(1,0)$ -type symbols on \mathbb{R}^{2d} . (See Section 4.5, 4.6 in [25].) In this way, a toroidal PDO or FSO can be rewritten into a periodic Euclidean PDO or FIO, which is of a globally defined integral form on $T^*\mathbb{T}^d$. In this section, we first introduce the notions of periodic Euclidean PDOs and FIOs, together with the rules of symbol calculus, the Egorov's theorems and the L^2 -continuity theorems. In Section 2.2.7, we explain how to identify toroidal PDOs or FSOs with periodic Euclidean PDOs or FIOs respectively.

³ In this paper, the tensor product of two Banach or Hilbert spaces always refers to the metric space completion of their algebraic tensor product.

2.2.1. *Periodic Euclidean symbols.* Functions on the torus \mathbb{T}^d or on its cotangent bundle $T^*\mathbb{T}^d$ can be considered in the following two different ways.

On one hand, we identify \mathbb{T}^d with the hypercube $[0, 1]^d \subset \mathbb{R}^d$. A function $\varphi \in C(\mathbb{T}^d)$ may thus be thought as a continuous function on $[0, 1]^d$ such that

$$\lim_{x_j \rightarrow 1^-} \varphi(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d) = \varphi(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_d)$$

for all $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d \in [0, 1]$ and $j = 1, 2, \dots, d$. We extend $\varphi \in C(\mathbb{T}^d)$ as an L^2 -function on \mathbb{R}^d by setting $\varphi|_{\mathbb{T}^d \setminus [0, 1]^d} \equiv 0$, and still denote it by φ . Similarly, a function $\varphi \in C(T^*\mathbb{T}^d)$ can be regarded as a continuous function on $[0, 1]^d \times \mathbb{R}^d$ with the boundary limit condition, and trivially extended to a function on \mathbb{R}^{2d} .

On the other hand, note that $\mathbb{T}^d \cong \mathbb{R}^d / \mathbb{Z}^d$. To lift a function on \mathbb{T}^d to a periodic function on \mathbb{R}^d , we introduce the periodization operator P given by

$$(2.7) \quad (P\varphi)(x) := \sum_{\mathbf{k} \in \mathbb{Z}^d} \varphi(x + \mathbf{k}), \quad x \in \mathbb{R}^d,$$

where φ is a function on \mathbb{R}^d such that the above sum converges absolutely. It is clear that $P\varphi$ is a 1-periodic function on \mathbb{R}^d . In particular, $P\varphi \in C(\mathbb{R}^d)$ is well-defined for any $\varphi \in C(\mathbb{T}^d)$. Similarly, we can define periodization operator P for suitable functions on \mathbb{R}^{2d} , i.e.,

$$(2.8) \quad (P\varphi)(x, \xi) := \sum_{\mathbf{k} \in \mathbb{Z}^d} \varphi(x + \mathbf{k}, \xi), \quad (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

as long as the above sum converges absolutely.

Denote $\mathbb{N}_0^d = \mathbb{N} \cup \{0\}$.

Definition 2.3. A complex-valued function $a \in C^\infty(T^*\mathbb{T}^d)$ is called a (periodic Euclidean) symbol of order $m \in \mathbb{R}$ on $T^*\mathbb{T}^d$ if

$$(2.9) \quad \mathcal{N}_{\alpha\beta, m}(a) := \sup_{(x, \xi) \in T^*\mathbb{T}^d} \frac{|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)|}{\langle \xi \rangle^{m-|\beta|}} < \infty$$

for any $\alpha, \beta \in \mathbb{N}_0^d$, where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. We denote the space of (periodic Euclidean) symbols of order m by S^m , which is short for $S^m(T^*\mathbb{T}^d)$.

The topology on the space S^m is generated by the seminorms $\{\mathcal{N}_{\alpha\beta, m}(\cdot)\}_{\alpha, \beta \in \mathbb{N}_0^d}$. For any $k \in \mathbb{N}_0$, we denote $\mathcal{N}_{k, m}(a) = \sup_{|\alpha| + |\beta| \leq k} \mathcal{N}_{\alpha\beta, m}(a)$. We often write $\mathcal{N}_k(a)$ for short if the order of a is clear.

Note that if $a \in S^m$ and $b \in S^{m'}$, then $a + b \in S^{\max\{m, m'\}}$, and $ab \in S^{m+m'}$. Also, for any $a \in S^m$ and $\alpha \in \mathbb{N}_0^d$, $\partial_x^\alpha a \in S^m$ and $\partial_\xi^\alpha a \in S^{m-|\alpha|}$. For these operations, we have corresponding seminorm relations, for instance, $\mathcal{N}_k(\partial_x^\alpha a) \leq \mathcal{N}_{k+|\alpha|}(a)$.

Remark 2.4. A function $a \in C^\infty(\mathbb{R}^{2d})$ is a Euclidean symbol of order m if (2.9) holds with the supremum taken over \mathbb{R}^{2d} . It is easy to see that the periodization Pa for a periodic Euclidean symbol $a \in S^m$ is a Euclidean symbol on \mathbb{R}^{2d} .

2.2.2. *Pseudo-differential operators.* In this subsection and the next one, we introduce the periodic Euclidean PDOs and FIOs on the torus. Since they are what we mostly use in this paper, we shall omit “periodic Euclidean” and simply call them PDOs or FIOs.

Let $\hbar \in (0, 1]$. In the general theory of semiclassical analysis, $\hbar \ll 1$ is the Planck’s constant parametrizing the whole family of symbol functions and thus

the symbol calculus for corresponding semiclassical pseudo-differential operators. We emphasize that the operators we study are quite particular: when regarded as classical PDOs or FIOs, the \hbar -scaled PDOs (or FIOs) all have symbols (or amplitudes) of the form $a(x, \hbar\xi)$ for some $a \in S^m$, and the \hbar -scaled FIOs all have phase functions of the form $\frac{1}{\hbar}S(x, \hbar\xi)$ for some $S \in S^1$. Consequently, the parameter \hbar merely serves as a scaling factor in the ξ -direction and would not affect the pseudo-differential calculus in a uniform scheme. In what follows, we will only introduce such restrictive semiclassical analysis.

Definition 2.5. *Given a symbol $a \in S^m$, the linear operator $\text{Op}_\hbar(a) : \mathcal{D}(\mathbb{T}^d) \rightarrow \mathcal{D}(\mathbb{T}^d)$ defined by*

$$(2.10) \quad \begin{aligned} \text{Op}_\hbar(a)\varphi(x) &= \int_{T^*\mathbb{T}^d} a(x, \xi) e^{i2\pi \frac{\xi}{\hbar} \cdot (x-y)} \varphi(y) dy d\left(\frac{\xi}{\hbar}\right) \\ &= \int_{T^*\mathbb{T}^d} a(x, \hbar\xi) e^{i2\pi \xi \cdot (x-y)} \varphi(y) dy d\xi \end{aligned}$$

is called a (periodic Euclidean) \hbar -scaled pseudo-differential operator (PDO) of order m corresponding to the symbol $a \in S^m$. We denote the space of \hbar -scaled PDOs of order m by $\text{Op}_\hbar S^m$.

Remark 2.6. *The integral in (2.10) is understood as over $T^*\mathbb{T}^d \cong [0, 1)^d \times \mathbb{R}^d$. Since $a(x, \xi)$ is only supported on $[0, 1)^d \times \mathbb{R}^d$, we can rewrite the integral as over \mathbb{R}^{2d} . In other words, the periodic Euclidean PDO $\text{Op}_\hbar(a)$ on \mathbb{T}^d can be considered as a Euclidean PDO $\text{Op}_\hbar^{\text{Eu}}(a)$ defined on \mathbb{R}^d , while the symbol a is smooth except at the boundary of $[0, 1)^d$, which is a set of Lebesgue measure zero.*

The formula with $\hbar = 1$ in (2.10) gives the definition of classical pseudo-differential operator $\text{Op}(a) = \text{Op}_1(a)$. We denote $\text{OPS}^m = \text{Op}_1 S^m$. In this way, the \hbar -scaled PDO with symbol $a \in S^m$ can be regarded as the classical PDO with symbol $a_\hbar \in S^m$, that is, $\text{Op}_\hbar(a) = \text{Op}(a_\hbar)$, where $a_\hbar(x, \xi) = a(x, \hbar\xi)$.

By standard duality argument, we extend $\text{Op}_\hbar(a) : \mathcal{D}'(\mathbb{T}^d) \rightarrow \mathcal{D}'(\mathbb{T}^d)$. Moreover, $\text{Op}_\hbar(a) : H^s(\mathbb{T}^d) \rightarrow H^{s-m}(\mathbb{T}^d)$ is a bounded operator if $a \in S^m$. Some properties about symbols and PDOs are stated in Section 2.2.4-2.2.6.

2.2.3. Fourier integral operators.

Definition 2.7. *A (periodic Euclidean) \hbar -scaled Fourier integral operator (FIO) $\Phi_\hbar : \mathcal{D}(\mathbb{T}^d) \rightarrow \mathcal{D}(\mathbb{T}^d)$ with amplitude function $a \in S^m$ and a real-valued phase function $S \in S^1$ is of the form*

$$(2.11) \quad \begin{aligned} \Phi_\hbar \varphi(x) &= \Phi_\hbar(a, S)\varphi(x) = \int_{T^*\mathbb{T}^d} a(x, \xi) e^{i2\pi \frac{1}{\hbar}[S(x, \xi) - y \cdot \xi]} \varphi(y) dy d\left(\frac{\xi}{\hbar}\right) \\ &= \int_{T^*\mathbb{T}^d} a(x, \hbar\xi) e^{i2\pi [\frac{1}{\hbar}S(x, \hbar\xi) - y \cdot \xi]} \varphi(y) dy d\xi, \end{aligned}$$

where the phase function $S(x, \xi)$ satisfies the following conditions:

- (1) there are $c_1, c_2 > 0$ such that $\left| \frac{\partial S(x, \xi)}{\partial x} \right| \geq c_1 |\xi|$ for all (x, ξ) with $|\xi| \geq c_2$;
- (2) $S(x, \xi)$ is strongly non-degenerate, i.e., there is $c_3 > 0$ such that

$$\left| \det \left(\frac{\partial^2 S(x, \xi)}{\partial x \partial \xi} \right) \right| \geq c_3 \quad \text{for any } (x, \xi) \in T^*\mathbb{R}^d.$$

Note that the *classical Fourier integral operator* $\Phi = \Phi_1$ is the one with $\hbar = 1$. Then the \hbar -scaled FIO with amplitude a and phase S can then be regarded as a classical FIO with amplitude a_\hbar and phase $S_\hbar(x, \xi) = \frac{1}{\hbar}S(x, \hbar\xi)$.

Remark 2.8.

(i) Hörmander's definition of phase functions usually assumes the homogeneity of degree one in ξ . Following Egorov [10], we replace the homogeneity by that $S \in S^1$ and Condition (1).

(ii) If we take $S(x, \xi) = x \cdot \xi$, then $\Phi_\hbar(a, S)$ becomes an \hbar -scaled pseudo-differential operator with the symbol a .

By standard duality argument, we can extend $\Phi_\hbar : \mathcal{D}'(\mathbb{T}^d) \rightarrow \mathcal{D}'(\mathbb{T}^d)$. Further, $\Phi_\hbar : H^s(\mathbb{T}^d) \rightarrow H^{s-m}(\mathbb{T}^d)$ is a bounded operator if its amplitude $a \in S^m$.

Definition 2.9. The canonical transformation associated to an \hbar -scaled FIO with phase S is the transformation $\mathcal{F}_\hbar : (x, \hbar\xi) \mapsto (y, \hbar\eta)$ given by

$$(2.12) \quad y = \frac{\partial S(x, \eta)}{\partial \eta}, \quad \xi = \frac{\partial S(x, \eta)}{\partial x}.$$

In other words, the phase function S serves as the generating function of the canonical transformation.

In the classical case when $\hbar = 1$, we write $\mathcal{F} = \mathcal{F}_1$.

2.2.4. The symbol calculus. If $m < m'$, then $S^m \subset S^{m'}$ and $\text{OP}_\hbar S^m \subset \text{OP}_\hbar S^{m'}$. Set $S^{-\infty} = \bigcap_{m \in \mathbb{R}} S^m$. If $a \in S^{-\infty}$, then $\text{Op}_\hbar(a)$ is a smoothing (and hence compact) operator.

Given two symbols $a, b \in S^m$, if the difference $a - b \in S^{m'}$ for some $-\infty \leq m' < m$, we write $a = b \pmod{S^{m'}}$. When \hbar is chosen, we shall denote $a = b \pmod{\hbar^{m-m'} S^{m'}}$ if $a - b = \hbar^{m-m'} r$ for some $r \in S^{m'}$.

Theorem 2.10. For classical PDOs, we have the following.

- (1) *Adjoint:* If $A \in \text{OPS}^m$ has a symbol a , then the adjoint operator $A^* \in \text{OPS}^m$ has a symbol $a^* = \bar{a} \pmod{S^{m-1}}$.
- (2) *Composition:* If $A \in \text{OPS}^m$ has a symbol a and $B \in \text{OPS}^{m'}$ has a symbol b , then the compositions $A \circ B \in \text{OPS}^{m+m'}$ has a symbol $a \# b = ab \pmod{S^{m+m'-1}}$.
- (3) *Inverse:* If $A \in \text{OPS}^m$ has a symbol $a > 0$ and is invertible, then $A^{-1} \in \text{OPS}^{-m}$ has a symbol $a^{-1} \pmod{S^{-m-1}}$.

Moreover, the \mathcal{N}_k -seminorm of all the remainders in the above modulo class only depends on the \mathcal{N}_{k+2} -seminorm of the original symbols.

Remark 2.11. The seminorm estimates of remainders can be easily seen from the proof of the symbol calculus. For instance, the symbol of the adjoint is given by

$$\begin{aligned} a^*(x, \xi) &= e^{-i2\pi \partial_\xi \cdot \partial_x} \overline{a(x, \xi)} := \int e^{-i2\pi(x-y) \cdot (\xi-\eta)} \overline{a(y, \eta)} dy d\eta \\ &= \sum_{|\alpha| < j} \frac{(-i2\pi)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a(x, \xi)} + r_j \end{aligned}$$

where r_j is an integral involving with only $\partial_\xi^\alpha \partial_x^\beta \bar{a}$ for $|\alpha| + |\beta| = 2j$. In particular, for $j = 1$, $a^* = \bar{a} + r_1$ and the seminorm $\mathcal{N}_k(r_1)$ only depends on $\mathcal{N}_{k+2}(a)$.

Recall that $\text{OP}_h(a)$ can be regarded as $\text{OP}(a_h)$, where $a_h(x, \xi) = a(x, h\xi)$. As a direct consequence of Theorem 2.10 and Remark 2.11, we have the following rules of the symbol calculus for the h -scaled PDOs.

Theorem 2.12.

- (1) *Adjoint:* If $A \in \text{OP}_h S^m$ has a symbol a , then the adjoint operator $A^* \in \text{OP}_h S^m$ has a symbol $a^* = \bar{a} \pmod{hS^{m-1}}$.
- (2) *Composition:* If $A \in \text{OP}_h S^m$ has a symbol a and $B \in \text{OP}_h S^{m'}$ has a symbol b , then the compositions $A \circ B \in \text{OP}_h S^{m+m'}$ has a symbol $a \# b = ab \pmod{hS^{m+m'-1}}$.
- (3) *Inverse:* If $A \in \text{OP}_h S^m$ has a symbol $a > 0$ and is invertible, then $A^{-1} \in \text{OP}_h S^{-m}$ has a symbol $a^{-1} \pmod{hS^{-m-1}}$.

Moreover, if hr is one of the remainders in the above modulo class, then the seminorm $\mathcal{N}_k(r)$ only depends on $\mathcal{N}_{k+2}(a)$.

2.2.5. Egorov's Theorem. Let Ω be an open domain in \mathbb{T}^d . We say that a symbol $a \in S^m$ is supported in $\Omega \times \mathbb{R}^d$ if $a(x, \xi) = 0$ for any $(x, \xi) \in (\mathbb{T}^d \setminus \Omega) \times \mathbb{R}^d$. The class of such symbols is denoted by $S^m(\Omega \times \mathbb{R}^d)$.

We first state the original version of classical Egorov's theorem in [10] for the invertible case.

Theorem 2.13. Let $A \in \text{OPS}^m$ with symbol $a \in S^m(\Omega \times \mathbb{R}^d)$, and Φ be a classical FIO with amplitude $b \in S^0$ and phase S . Let $\mathcal{F}(x, \xi) = (y, \eta)$ be the canonical transformation associated to Φ (as defined in (2.12) with $h = 1$), and Ω' be the image of $\Omega \times \mathbb{R}^d$ under the first d components of \mathcal{F} . We assume that $\mathcal{F} : \Omega \times \mathbb{R}^d \rightarrow \Omega' \times \mathbb{R}^d$ is a bijective map. Then the operator $\Phi^* A \Phi \in \text{OPS}^m$ has a symbol $\tilde{a} \in S^m(\Omega' \times \mathbb{R}^d)$ such that

$$\tilde{a}(y, \eta) = \tilde{a}(\mathcal{F}(x, \xi)) = a(x, \xi) |b(x, \xi)|^2 \left| \det \left(\frac{\partial^2 S}{\partial x \partial \xi} \right) \right|^{-1} \pmod{S^{m-1}}.$$

Remark 2.14. From the proof in [10], it is easy to see that the \mathcal{N}_k -seminorm of the remainder in the above modulo class only relies on the \mathcal{N}_{k+2} -seminorms of a, b and the \mathcal{N}_{k+4} -seminorm of S .

For our purpose, we need the following version of Egorov's theorem.

Theorem 2.15. Let $A \in \text{OPS}^m$ with symbol $a \in S^m$, and Φ be a classical FIO with amplitude $b \in S^0$ and phase S . Let $\mathcal{F}(x, \xi) = (y, \eta)$ be the canonical transformation associated to Φ . We assume that \mathcal{F} is a surjective local diffeomorphism of $T^*\mathbb{T}^d$ with finite inverse branches. Moreover, for each $x \in \mathbb{T}^d$, the map $\xi \mapsto \mathcal{F}(x, \xi)$ is bijective. Then the operator $\Phi^* A \Phi \in \text{OPS}^m$ has a symbol \tilde{a} such that

$$(2.13) \quad \tilde{a}(y, \eta) = \sum_{\mathcal{F}(x, \xi) = (y, \eta)} a(x, \xi) |b(x, \xi)|^2 \left| \det \left(\frac{\partial^2 S}{\partial x \partial \xi} \right) \right|^{-1} \pmod{S^{m-1}}.$$

Moreover, the \mathcal{N}_k -seminorm of the remainder in the above modulo class only relies on the \mathcal{N}_{k+2} -seminorms of a, b and the \mathcal{N}_{k+4} -seminorm of S .

Proof. By the properties of the canonical map \mathcal{F} , we can choose an finite open cover $\{\Omega_i\}$ of \mathbb{T}^d such that each $\Omega_i \times \mathbb{R}^d$ is strictly inside an inverse branch of \mathcal{F} . By partition of unity, there are $\chi_i \in C_0^\infty(\Omega_i; [0, 1])$ such that $\sum_i \chi_i = 1$. We define

symbols $a_i \in S^m(\Omega_i \times \mathbb{R}^d)$ by $a_i(x, \xi) = \chi_i(x)a(x, \xi)$, and set $A_i = \text{Op}(a_i)$. By Theorem 2.13, each $\Phi^* A_i \Phi \in \text{OPS}^m$ has a symbol \tilde{a}_i such that

$$\tilde{a}_i(y, \eta) = \chi_i(x)a(x, \xi)|b(x, \xi)|^2 \left| \det \left(\frac{\partial^2 S}{\partial x \partial \xi} \right) \right|^{-1} \pmod{S^{m-1}}$$

for any $(y, \eta) \in \mathcal{F}(\Omega_i \times \mathbb{R}^d)$ with the only pre-image (x, ξ) in $\Omega_i \times \mathbb{R}^d$, and $\tilde{a}_i(y, \eta) = 0$ if $(y, \eta) \notin \mathcal{F}(\Omega_i \times \mathbb{R}^d)$. Therefore,

$$\Phi^* A \Phi = \Phi^* \left(\sum_i A_i \right) \Phi = \sum_i \Phi^* A_i \Phi \in \text{OPS}^m,$$

and its symbol $\tilde{a}(y, \eta)$ is given by

$$\begin{aligned} \sum_i \tilde{a}_i(y, \eta) &= \sum_{\mathcal{F}(x, \xi) = (y, \eta)} \sum_{i: x \in \Omega_i} \chi_i(x)a(x, \xi)|b(x, \xi)|^2 \left| \det \left(\frac{\partial^2 S}{\partial x \partial \xi} \right) \right|^{-1} \pmod{S^{m-1}} \\ &= \sum_{\mathcal{F}(x, \xi) = (y, \eta)} a(x, \xi)|b(x, \xi)|^2 \left| \det \left(\frac{\partial^2 S}{\partial x \partial \xi} \right) \right|^{-1} \left(\sum_{i: x \in \Omega_i} \chi_i(x) \right) \pmod{S^{m-1}} \\ &= \sum_{\mathcal{F}(x, \xi) = (y, \eta)} a(x, \xi)|b(x, \xi)|^2 \left| \det \left(\frac{\partial^2 S}{\partial x \partial \xi} \right) \right|^{-1} \pmod{S^{m-1}}. \end{aligned}$$

The seminorm dependence of the remainder is straightforward by Remark 2.14. \square

Remark 2.16. We can easily adapt the proof of Theorem 2.13 and 2.15 in the \hbar -scaled situation and show that if $A \in \text{Op}_\hbar S^m$ has a symbol a and Φ_\hbar is the \hbar -scaled FIO with amplitude $b \in S^0$ and phase S , then the symbol of $\Phi_\hbar^* A \Phi_\hbar \in \text{Op}_\hbar S^m$ is still given by (2.13) but with $\mathcal{F}(x, \xi) = (y, \eta)$ replaced by $\mathcal{F}_\hbar(x, \hbar\xi) = (y, \hbar\eta)$, and $\pmod{S^{m-1}}$ replaced by $\pmod{\hbar S^{m-1}}$. Moreover, if $\hbar r$ is the remainder in the modulo class, then $\mathcal{N}_k(r)$ only depends on the \mathcal{N}_{k+2} -seminorms of a, b and the \mathcal{N}_{k+4} -seminorm of S .

2.2.6. L^2 -Continuity. The following result is the classical Calderon-Vaillancourt theorem (see Theorem 2.8.1 in [19] for instance).

Theorem 2.17. Let $a(x, \xi) \in S^0$, then $\text{Op}(a) : L^2(\mathbb{T}^d) \rightarrow L^2(\mathbb{T}^d)$ is a bounded operator such that

$$\|\text{Op}(a)\|_{L^2 \rightarrow L^2} \leq M_1 \|a\|_{C^{k_1}} \leq M_1 \mathcal{N}_{k_1}(a),$$

for some $M_1 > 0$ and $k_1 \in \mathbb{N}$ that only depend on the dimension d .

To get finer L^2 -norm estimates, we first state a version of L^2 -continuity for a classical PDO of order 0 established in [14].

Theorem 2.18. If $a(x, \xi) \in S^0$, then $\text{Op}(a) : L^2(\mathbb{T}^d) \rightarrow L^2(\mathbb{T}^d)$ is a bounded operator. Moreover, for any $\varepsilon > 0$, there is a decomposition

$$\text{Op}(a) = K(\varepsilon) + R(\varepsilon)$$

such that $K(\varepsilon) : L^2(\mathbb{T}^d) \rightarrow L^2(\mathbb{T}^d)$ is a compact operator and

$$\|R(\varepsilon)\|_{L^2 \rightarrow L^2} \leq \sup_x \limsup_{|\xi| \rightarrow \infty} |a(x, \xi)| + \varepsilon = \sup_x \limsup_{|\xi| \rightarrow \infty} |a_0(x, \xi)| + \varepsilon,$$

where $a_0 \in S^0$ is such that $a = a_0 \pmod{S^{-1}}$.

For an \hbar -scaled PDO of order 0, we need a version of Calderon-Vaillancourt theorem, which applies not only for $\hbar \rightarrow 0$ but for arbitrary $\hbar \in (0, 1]$. See similar statements in [29], Theorem 4.23 or Theorem 5.1 in the formulation of Weyl quantization.

Theorem 2.19. *If $a(x, \xi) \in S^0$, then $\text{Op}_\hbar(a) : L^2(\mathbb{T}^d) \rightarrow L^2(\mathbb{T}^d)$ is a bounded operator. Moreover, if $a = a_0 + \hbar r$ for some $a_0 \in S^0$ and $r \in S^{-1}$, then*

$$\|\text{Op}_\hbar(a)\|_{L^2 \rightarrow L^2} \leq \sup_{(x, \xi) \in T^*\mathbb{T}^d} |a_0(x, \xi)| + \hbar C_{k_2}(a_0, r),$$

where the constant $C_{k_2}(a_0, r)$ only depends on the \mathcal{N}_{k_2} -seminorms of a_0 and r , for some $k_2 \in \mathbb{N}$ that only depends on the dimension d .

Proof. Recall that $\text{Op}_\hbar(a) = \text{Op}(a_\hbar)$, where $a_\hbar(x, \xi) = a(x, \hbar\xi) \in S^0$, by Theorem 2.17, $\text{Op}_\hbar(a)$ is a bounded operator on $L^2(\mathbb{T}^d)$. For the operator norm estimate, we mimic the proof in Section 7.5 of [27], which is originally due to Hörmander “square root trick”.

Pick any $M > \sup_{(x, \xi) \in T^*\mathbb{T}^d} |a_0(x, \xi)|$, and set $b = \sqrt{M^2 - |a_0|^2} \in S^0$. By Theorem 2.12, the operator $\text{Op}_\hbar(a)^* \text{Op}_\hbar(a) \in \text{Op}_\hbar S^0$ has a symbol

$$\begin{aligned} a^* \# a &= \bar{a}a \pmod{\hbar S^{-1}} = |a_0|^2 + 2\hbar \Re(a_0 r) + \hbar^2 |r|^2 \pmod{\hbar S^{-1}} \\ &= M^2 - b^2 \pmod{\hbar S^{-1}}, \end{aligned}$$

that is, $a^* \# a = M^2 - b^2 + \hbar r_1$, for some $r_1 \in S^{-1}$. Therefore, for any $\varphi \in L^2(\mathbb{T}^d)$,

$$\begin{aligned} \|\text{Op}_\hbar(a)\varphi\|_{L^2}^2 &= \langle \text{Op}_\hbar(a)^* \text{Op}_\hbar(a)\varphi, \varphi \rangle_{L^2} \\ &= M^2 \|\varphi\|_{L^2}^2 - \|\text{Op}_\hbar(b)\varphi\|_{L^2}^2 + \hbar \langle \text{Op}_\hbar(r_1)\varphi, \varphi \rangle_{L^2} \\ &\leq (M^2 + \hbar \|\text{Op}_\hbar(r_1)\|_{L^2 \rightarrow L^2}) \|\varphi\|_{L^2}^2 \\ &\leq (M + \hbar \cdot \frac{\|\text{Op}_\hbar(r_1)\|_{L^2 \rightarrow L^2}}{2M})^2 \|\varphi\|_{L^2}^2. \end{aligned}$$

It remains to show that $\frac{\|\text{Op}_\hbar(r_1)\|_{L^2 \rightarrow L^2}}{2M}$ has an upper bound that is related to a_0 and r . Indeed, by Theorem 2.17,

$$\|\text{Op}_\hbar(r_1)\|_{L^2 \rightarrow L^2} = \|\text{Op}((r_1)_\hbar)\|_{L^2 \rightarrow L^2} \leq M_1 \mathcal{N}_{k_1}(r_1(x, \hbar\xi)) \leq M_1 \mathcal{N}_{k_1}(r_1).$$

By the construction of r_1 , we have that $\mathcal{N}_{k_1}(r_1)$ depends only on the \mathcal{N}_{k_1+2} -seminorms of a, a_0, r , and thus only of a_0, r , since $\mathcal{N}_{k_1+2}(a) = \mathcal{N}_{k_1+2}(a_0 + \hbar r) \leq \mathcal{N}_{k_1+2}(a_0) + \mathcal{N}_{k_1+2}(r)$. In other words, let $k_2 = k_1 + 2$, then

$$\frac{\|\text{Op}_\hbar(r)\|_{L^2 \rightarrow L^2}}{2M} \leq \frac{M_1 \mathcal{N}_{k_1}(r_1)}{2 \sup_{(x, \xi)} |a_0(x, \xi)|} \leq C_{k_2}(a_0, r),$$

where $C_{k_2}(a_0, r)$ is a constant that only depends on the \mathcal{N}_{k_2} -seminorms of a_0 and r . This finishes the proof of the theorem. \square

2.2.7. Equivalence between Toroidal quantization and periodic Euclidean quantization. In [25], a toroidal symbol on $\mathbb{T}^d \times \mathbb{Z}^d$ is defined in a similar way as that in Definition 2.3, while the differential ∂_ξ is replaced by the difference Δ_ξ . Equivalently, a function $a_{\text{tor}} \in C^\infty(\mathbb{T}^d \times \mathbb{Z}^d)$ is a toroidal symbol of order $m \in \mathbb{R}$ if there is a periodic Euclidean symbol $a \in S^m$ such that $a|_{\mathbb{T}^d \times \mathbb{Z}^d} = a_{\text{tor}}$. We denote the

space of toroidal symbols of order m by $S^m(\mathbb{T}^d \times \mathbb{Z}^d)$. The corresponding \hbar -scaled toroidal PDO $\text{Op}_\hbar^{\text{tor}}(a_{\text{tor}})$ for $a_{\text{tor}} \in S^m(\mathbb{T}^d \times \mathbb{Z}^d)$ is defined by

$$\text{Op}_\hbar^{\text{tor}}(a_{\text{tor}})\varphi(x) := \sum_{\xi \in \hbar\mathbb{Z}^d} \int_{\mathbb{T}^d} a_{\text{tor}}(x, \xi) e^{i2\pi \frac{\xi}{\hbar} \cdot (x-y)} \varphi(y) dy, \quad \varphi \in \mathcal{D}(\mathbb{T}^d),$$

which is similar to the periodic Euclidean PDO given by Definition 2.5 but with a discrete measure over $\hbar\mathbb{Z}^d$ in the ξ -direction. Similarly, a function $S_{\text{tor}} \in C^\infty(\mathbb{T}^d \times \mathbb{Z}^d)$ is a toroidal phase if there is a periodic Euclidean phase S such that $S|_{\mathbb{T}^d \times \mathbb{Z}^d} = S_{\text{tor}}$. With toroidal amplitude a_{tor} and phase S_{tor} , we define the \hbar -scaled toroidal Fourier series operator (FSO) by

$$\begin{aligned} \Phi_\hbar^{\text{tor}} \varphi(x) &= \Phi_\hbar^{\text{tor}}(a_{\text{tor}}, S_{\text{tor}})\varphi(x) \\ &= \sum_{\xi \in \hbar\mathbb{Z}^d} \int_{\mathbb{T}^d} a_{\text{tor}}(x, \xi) e^{i2\pi \frac{1}{\hbar} \cdot [S_{\text{tor}}(x, \xi) - y \cdot \xi]} \varphi(y) dy, \quad \varphi \in \mathcal{D}(\mathbb{T}^d). \end{aligned}$$

Let us recall the Euclidean PDOs and FIOs on \mathbb{R}^{2d} as well. Given a Euclidean symbol $a \in S^m(\mathbb{R}^{2d})$, we define the \hbar -scaled Euclidean PDO $\text{Op}_\hbar^{\text{Eu}}(a) : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ by the formula (2.10) but with integral over \mathbb{R}^{2d} . Here $\mathcal{S}(\mathbb{R}^d)$ denotes the class of Schwartz functions on \mathbb{R}^d , i.e., for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta \varphi(x)| < \infty$ for any $\alpha, \beta \in \mathbb{N}_0^d$. Similarly, given amplitude a and phase S on \mathbb{R}^{2d} , we can define the \hbar -scaled Euclidean FIOs similar to (2.11) but with integral over \mathbb{R}^{2d} , denoted by $\Phi_\hbar^{\text{Eu}} = \Phi_\hbar^{\text{Eu}}(a, S)$.

To show that the toroidal quantization is indeed equivalent to the periodic Euclidean quantization that we have introduced in previous subsections, we need the periodization operator \mathbf{P} given by (2.7) and (2.8). (See Section 4.5, 4.6 in [25] for more details.) Let π_0 be the restriction of a function on \mathbb{R}^d onto $[0, 1]^d$. It turns out that $\mathbf{P}_0 = \pi_0 \circ P : \mathcal{S}(\mathbb{R}^d) \rightarrow C^\infty([0, 1]^d) \cong \mathcal{D}(\mathbb{T}^d)$ is surjective. Moreover, given $a_{\text{tor}} \in S^m(\mathbb{T}^d \times \mathbb{Z}^d)$ extendable to a periodic Euclidean symbol $a \in S^m = S^m(T^*\mathbb{T}^d)$, we have that $\mathbf{P}a \in S^m(\mathbb{R}^{2d})$, and

$$(2.14) \quad \text{Op}_\hbar^{\text{tor}}(a_{\text{tor}})(\varphi) = \mathbf{P}_0 \circ \text{Op}_\hbar^{\text{Eu}}(\mathbf{P}a)(\varphi) = \text{Op}_\hbar(a)(\varphi) + K_\hbar(a)(\varphi)$$

for any $\varphi \in C_0^\infty([0, 1]^d) = C_0^\infty(\mathbb{T}^d) \subset \mathcal{D}(\mathbb{T}^d)$, where $K_\hbar(a) \in \text{Op}_\hbar(S^{-\infty})$ only depends on a . Similarly, given toroidal amplitude $a_{\text{tor}} = a|_{\mathbb{T}^d \times \mathbb{Z}^d}$ extendable to a periodic Euclidean amplitude $a \in S^m$, and toroidal phase $S_{\text{tor}} = S|_{\mathbb{T}^d \times \mathbb{Z}^d}$ extendable to a periodic Euclidean phase S , we have that $\mathbf{P}a$ and $\mathbf{P}S$ are Euclidean amplitude and phase respectively, and

$$(2.15) \quad \Phi_\hbar^{\text{tor}}(a_{\text{tor}}, S_{\text{tor}})(\varphi) = \mathbf{P}_0 \circ \Phi_\hbar^{\text{Eu}}(\mathbf{P}a, \mathbf{P}S)(\varphi) = \Phi_\hbar(a, S)(\varphi) + K_\hbar(a, S)(\varphi)$$

for any $\varphi \in C_0^\infty([0, 1]^d) = C_0^\infty(\mathbb{T}^d) \subset \mathcal{D}(\mathbb{T}^d)$, where $K_\hbar(a, S)$ is an \hbar -scaled smoothing operator which only depends on a and S . To sum up, an \hbar -scaled toroidal PDO or FSO can be identified with an \hbar -scaled periodic Euclidean PDO or FIO up to an \hbar -scaled smoothing operator.⁴

Let us make a comment on the operator norm estimates of the smoothing operators in (2.14) and (2.15). The \hbar -scaled smoothing operator $K_\hbar(a)$ (or $K_\hbar(a, S)$) is a bounded operator from $H_\hbar^s(\mathbb{T}^d)$ to itself for any $s \in \mathbb{R}$, where $H_\hbar^s(\mathbb{T}^d)$ is the Sobolev space $H^s(\mathbb{T}^d)$ but endowed with the \hbar -scaled inner product $\langle \cdot, \cdot \rangle_{s, \hbar}$ (See

⁴ This identification is first true for action on $C_0^\infty(\mathbb{T}^d)$, by standard duality, it is also true for $(C_0^\infty(\mathbb{T}^d))' \supset \mathcal{D}'(\mathbb{T}^d)$. By the density of $C_0^\infty(\mathbb{T}^d)$ in $\mathcal{D}'(\mathbb{T}^d)$, the identification is true for action on $\mathcal{D}'(\mathbb{T}^d)$ and Sobolev spaces $H^s(\mathbb{T}^d)$ of any order s .

the definition in (2.5)). By stationary phase approximation (see e.g. Section 3.4, 3.5 in [29]), one has $\|K_h(a)|H_h^s(\mathbb{T}^d)\| = O(h^\infty)$ (or $\|K_h(a, S)|H_h^s(\mathbb{T}^d)\| = O(h^\infty)$). In particular, there is $k_3 = k_3(d, s) \in \mathbb{N}$ such that $\|K_h(a)|H_h^s(\mathbb{T}^d)\| \leq hC_{k_3}(a)$ (or $\|K_h(a, S)|H_h^s(\mathbb{T}^d)\| \leq hC_{k_3}(a, S)$), where $C_{k_3}(a)$ only depends on the seminorm $\mathcal{N}_{k_3}(a)$, and $C_{k_3}(a, S)$ only depends on the seminorms $\mathcal{N}_{k_3}(a)$ and $\mathcal{N}_{k_3+2}(S)$.

3. SPECTRAL GAP AND COBOUNDARY: PROOF OF THE THEOREMS

3.1. Decomposition of Koopman operator. In this subsection we decompose the Koopman operator $\hat{F} : \mathcal{W}^s \rightarrow \mathcal{W}^s$ according to fiberwise Fourier expansion, where $\mathcal{W}^s = H^s(\mathbb{T}^d) \otimes H^{-s}(\mathbb{T}^\ell)$ is given in (2.6). Recall that $s < 0$ is an arbitrary negative order.

Given $\phi \in \mathcal{W}^s$, we write the Fourier series expansion along \mathbb{T}^ℓ -direction as

$$\phi(x, y) = \sum_{\nu \in \mathbb{Z}^\ell} \phi_\nu(x) e^{i2\pi\nu \cdot y},$$

where the Fourier coefficients are defined by

$$\phi_\nu(x) = \int_{\mathbb{T}^\ell} \phi(x, y) e^{-i2\pi\nu \cdot y} dy \in H^s(\mathbb{T}^d), \quad \nu \in \mathbb{Z}^\ell.$$

It is clear that the family of functions $\{e^{i2\pi\nu \cdot y}\}_{\nu \in \mathbb{Z}^\ell}$ forms an orthogonal basis of $H^{-s}(\mathbb{T}^\ell)$, and $\|e^{i2\pi\nu \cdot y}\|_{H^{-s}(\mathbb{T}^\ell)} = \langle \nu \rangle^{-s}$. Therefore, the Hilbert inner product on \mathcal{W}^s is given by

$$\langle \phi^1, \phi^2 \rangle_{\mathcal{W}^s} = \sum_{\nu \in \mathbb{Z}^\ell} \langle \nu \rangle^{-2s} \langle \phi_\nu^1, \phi_\nu^2 \rangle_{H^s(\mathbb{T}^d)}, \quad \text{for any } \phi^1, \phi^2 \in \mathcal{W}^s.$$

We thus consider the $\langle \nu \rangle^{-1}$ -scaled s -inner product $\langle \cdot, \cdot \rangle_{s, \langle \nu \rangle^{-1}}$ on $H^s(\mathbb{T}^d)$, or $\langle \cdot, \cdot \rangle_{s, \nu}$ for short. That is, by (2.5), we have for $\varphi, \psi \in H^s(\mathbb{T}^d)$,

$$(3.1) \quad \langle \varphi, \psi \rangle_{s, \nu} = \langle \nu \rangle^{-2s} \langle \varphi, \psi \rangle_s = \sum_{\xi \in \mathbb{Z}^d} \langle \nu \rangle^{-2s} \langle \xi \rangle^{2s} \widehat{\varphi}(\xi) \overline{\widehat{\psi}(\xi)},$$

and we denote by $H_\nu^s(\mathbb{T}^d)$ for the space of s -order Sobolev functions on \mathbb{T}^d endowed with the new inner product $\langle \cdot, \cdot \rangle_{s, \nu}$. Note that $H_\nu^s(\mathbb{T}^d) = H^s(\mathbb{T}^d)$ as spaces of Sobolev functions, although they are equipped with different but equivalent inner products. In this way, we obtain an orthogonal decomposition

$$\mathcal{W}^s = H^s(\mathbb{T}^d) \otimes H^{-s}(\mathbb{T}^\ell) \cong \bigoplus_{\nu \in \mathbb{Z}^\ell} H_\nu^s(\mathbb{T}^d),$$

such that the inner product of two functions $\phi^j(x, y) = \sum_{\nu \in \mathbb{Z}^\ell} \phi_\nu^j(x) e^{i2\pi\nu \cdot y} \in \mathcal{W}^s$, $j = 1, 2$, is given by

$$\langle \phi^1, \phi^2 \rangle_{\mathcal{W}^s} = \sum_{\nu \in \mathbb{Z}^\ell} \langle \phi_\nu^1, \phi_\nu^2 \rangle_{H_\nu^s(\mathbb{T}^d)}.$$

Also this decomposition is \hat{F} -invariant, since for each Fourier mode $\nu \in \mathbb{Z}^\ell$,

$$\hat{F}(\phi_\nu(x) e^{i2\pi\nu \cdot y}) = [\phi_\nu(Tx) e^{i2\pi\nu \cdot \tau(x)}] e^{i2\pi\nu \cdot y},$$

and it can be shown that $\phi_{\nu}(Tx)e^{i2\pi\nu\cdot\tau(x)} \in H^s(\mathbb{T}^d) \cong H_{\nu}^s(\mathbb{T}^d)$.⁵ Correspondingly, we decompose $\hat{F} = \bigoplus_{\nu \in \mathbb{Z}^d} \hat{F}_{\nu}$, where each $\hat{F}_{\nu} \cong \hat{F}|_{H_{\nu}^s(\mathbb{T}^d)}$ acts by

$$(3.2) \quad \hat{F}_{\nu}\varphi(x) = \varphi(Tx)e^{i2\pi\nu\cdot\tau(x)}, \quad \varphi \in H_{\nu}^s(\mathbb{T}^d).$$

Using the fact that $H_{\nu}^s(\mathbb{T}^d)$ is the dual space of $H_{\nu}^{-s}(\mathbb{T}^d)$, we get the dual operator

$$(3.3) \quad \hat{F}'_{\nu}\psi(x) = \sum_{Ty=x} \frac{e^{i2\pi\nu\cdot\tau(y)}}{|\text{Jac}(T)(y)|} \psi(y), \quad \psi \in H_{\nu}^{-s}(\mathbb{T}^d).$$

In other words, $\hat{F}'_{\nu}|_{H_{\nu}^{-s}(\mathbb{T}^d)}$ is the RPF transfer operator over $T : \mathbb{T}^d \rightarrow \mathbb{T}^d$ for the complex potential function $-\log|\text{Jac}(T)| + i2\pi\nu \cdot \tau$. In the case when $\nu = \mathbf{0}$, we have that $\hat{F}'_{\mathbf{0}}h = h$, that is, the density function $h(x)$ of the smooth invariant measure μ w.r.t. dx is provided by the eigenvector corresponding to the leading simple eigenvalue 1 of $\hat{F}'_{\mathbf{0}}$. See [24] for more details.

We shall use the fact $\hat{F}'_{\mathbf{0}}h = h$ in the following particular form:

$$\sum_{Ty=x} \mathcal{A}(y) = 1, \quad \text{for all } x \in \mathbb{T}^d,$$

where

$$(3.4) \quad \mathcal{A}(y) = \frac{1}{|\text{Jac}(T)(y)|} \frac{h(y)}{h(Ty)}.$$

Similarly, we have for all $n \in \mathbb{N}$,

$$\sum_{T^n y=x} \mathcal{A}_n(y) = 1, \quad \text{for all } x \in \mathbb{T}^d,$$

where

$$(3.5) \quad \mathcal{A}_n(y) = \frac{1}{|\text{Jac}(T^n)(y)|} \frac{h(y)}{h(T^n y)}.$$

3.2. Spectral gap. Recall that the notion of spectral gap is given right before Theorem 2 is stated (see (1.4)).

According to the decomposition of $\hat{F} : \mathcal{W}^s \rightarrow \mathcal{W}^s$, the spectral gap property follows from the following propositions. The proof of the propositions will be given in the next section, using the semiclassical analysis theory.

Proposition 3.1. *Let $s < 0$. There are $C_1 > 0$ and $\rho_1 \in (0, 1)$ such that for any $\nu \in \mathbb{Z}^d$, $\hat{F}_{\nu} : H_{\nu}^s(\mathbb{T}^d) \rightarrow H_{\nu}^s(\mathbb{T}^d)$ can be written as*

$$\hat{F}_{\nu} = K_{\nu} + R_{\nu},$$

where K_{ν} is a compact operator and

$$(3.6) \quad \|R_{\nu}^n|_{H_{\nu}^s(\mathbb{T}^d)}\| \leq C_1 \rho_1^n, \quad n \in \mathbb{N}.$$

Proposition 3.2. *Let $s < 0$ and assume that τ is not an essential coboundary. There are $C_2 > 0$, $\rho_2 \in (0, 1)$ and $\nu_1 > 0$ such that for any $\nu \in \mathbb{Z}^d$ with $|\nu| \geq \nu_1$,*

$$\|\hat{F}_{\nu}^n|_{H_{\nu}^s(\mathbb{T}^d)}\| \leq C_2 \rho_2^n, \quad n \in \mathbb{N}.$$

⁵ This fact is easy to show for $s \in \mathbb{N} \cup \{0\}$, and hence is also true when s is a negative integer by duality. For the general case, treat H^s as the interpolation between $H^{\lfloor s \rfloor}$ and $H^{\lfloor s \rfloor + 1}$. See Section 4.2 in [26] for details.

Remark 3.3.

(i) The quasi-compactness property is well known for Ruelle-Perron-Frobenius transfer operator on Hölder function spaces over expanding maps. Proposition 3.1 can be regarded as its dual version. The estimate in (3.6) shows that the essential spectral radius of \hat{F}_ν is no more than ρ_1 . See (4.7) for the definition of ρ_1 , which depends on the Sobolev order s and the minimal expansion rate γ given by (1.1);

(ii) Proposition 3.2 shows that the operator \hat{F}_ν is essentially a contraction when the Fourier mode ν is very large, and the spectral radius of \hat{F}_ν is no more than ρ_2 . See (4.15) for the definition of ρ_2 .

3.3. Proof of Theorem 2. Recall that the space $\mathcal{W}^s = H^s(\mathbb{T}^d) \otimes H^{-s}(\mathbb{T}^\ell)$ is defined in (2.6), where $s < 0$.

Lemma 3.4. *The spectral radius $\text{Sp}(\hat{F}_\nu|H_\nu^s(\mathbb{T}^d)) \leq 1$ for $\nu \in \mathbb{Z}^\ell$.*

Proof. The proof is similar as in [14], as we sketch here.

Choose $\rho_3 \in (\rho_1, 1)$, where ρ_1 is given in Proposition 3.1. Then we can rewrite

$$\hat{F}_\nu = K_\nu + R_\nu = (K_\nu^1 + K_\nu^2) + R_\nu = K_\nu^1 + (K_\nu^2 + R_\nu) = K_\nu^1 + R'_\nu$$

such that the spectral radius of R'_ν is less than ρ_3 . This can be done by defining K_ν^1 and K_ν^2 to be the spectral projection of K_ν outside and inside the circle $\{z : |z| = \rho_3\}$ respectively. Note that K_ν^1 has finite rank since K_ν is compact. To prove that $\text{Sp}(\hat{F}_\nu|H_\nu^s(\mathbb{T}^d)) \leq 1$ for $\nu \in \mathbb{Z}^\ell$, it is sufficient to show that all eigenvalues of K_ν^1 are of modulus no more than 1.

The general Jordan decomposition of K_ν^1 can be written

$$K_\nu^1 = \sum_{i=1}^k \left(\lambda_i \sum_{j=1}^{d_i} v_{ij} \otimes w_{ij} + \sum_{j=1}^{d_i-1} v_{ij} \otimes w_{i(j+1)} \right)$$

where d_i is the dimension of the Jordan block associated with the eigenvalue λ_i , with $v_{ij} \in H_\nu^s(\mathbb{T}^d)$ and $w_{ij} \in H_\nu^{-s}(\mathbb{T}^d)$. We arrange eigenvalues such that $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_k|$.

Now if $|\lambda_1| > 1$, we can choose $\varphi, \psi \in \mathcal{D}(\mathbb{T}^d)$ such that $v_{11}(\psi) \neq 0$ and $w_{11}(\varphi) \neq 0$ since $\mathcal{D}(\mathbb{T}^d)$ is dense in both $H_\nu^s(\mathbb{T}^d)$ and $H_\nu^{-s}(\mathbb{T}^d)$. On one hand,

$$\left| (\psi, \hat{F}_\nu^n \varphi)_{H_\nu^{-s}, H_\nu^s} \right| = \left| \int_{\mathbb{T}^d} \psi \hat{F}_\nu^n \varphi dx \right| \leq \int |\psi| |\varphi| \circ T^n dx \leq |\psi|_{C^0} |\varphi|_{C^0}.$$

On the other hand,

$$\left| (\psi, \hat{F}_\nu^n \varphi)_{H_\nu^{-s}, H_\nu^s} \right| \geq \left| (\psi, (K_\nu^1)^n \varphi)_{H_\nu^{-s}, H_\nu^s} \right| - \left| (\psi, (R'_\nu)^n \varphi)_{H_\nu^{-s}, H_\nu^s} \right|.$$

The second term converges to 0 since $\|(R'_\nu)^n|H_\nu^s(\mathbb{T}^d)\| = O(\rho_3^n)$, while the first term

$$\left| (\psi, (K_\nu^1)^n \varphi)_{H_\nu^{-s}, H_\nu^s} \right| = \left| \sum_{i=1}^k \sum_{r=0}^{\min(n, d_i-1)} \binom{n}{r} \lambda_i^{n-r} \sum_{j=1}^{d_i-r} v_{ij}(\psi) w_{i(j+r)}(\varphi) \right|$$

has a leading growth $|\lambda_1|^n |v_{11}(\psi)| |w_{11}(\varphi)| \rightarrow \infty$ as $n \rightarrow \infty$, which is a contradiction. Therefore, all eigenvalues of K_ν^1 are of modulus no more than 1. \square

Lemma 3.5. *If τ is not an essential coboundary over T , then the spectral radius $\text{Sp}(\hat{F}_\nu|H_\nu^s(\mathbb{T}^d)) < 1$ for $\nu \in \mathbb{Z}^\ell \setminus \{\mathbf{0}\}$. Moreover, 1 is the only eigenvalue of $\hat{F}_\mathbf{0}$ on the unit circle, which is simple with eigenspace of constant functions.*

Proof. Note that the essential spectral radius of $\hat{F}_\nu|H_\nu^s(\mathbb{T}^d)$ is no more than ρ_1 by (3.6). For any $\rho_3 \in (\rho_1, 1)$, by Lemma 3.4, the spectrum of $\hat{F}_\nu|H_\nu^s(\mathbb{T}^d)$ in $\{z \in \mathbb{C} : \rho_3 \leq |z| \leq 1\}$ consists of finitely many isolated eigenvalues of finite multiplicity. Consequently, the spectral radius $\text{Sp}(\hat{F}_\nu|H_\nu^s(\mathbb{T}^d))$ equals to the largest modulus of its eigenvalues.

Let λ be an eigenvalue of \hat{F}_ν with modulus 1, and $\varphi \in H_\nu^s(\mathbb{T}^d) \subset H_\nu^{s-\frac{d}{2}-1}(\mathbb{T}^d)$ be a corresponding eigenvector such that $\hat{F}_\nu\varphi = \lambda\varphi$. To prove this lemma, it is sufficient to show that $\nu = \mathbf{0}$, $\lambda = 1$, and φ is a constant function.

The following argument is essentially due to Pollicott [22]. By duality, there is $\psi \in H_\nu^{-s+\frac{d}{2}+1}(\mathbb{T}^d) \subset C(\mathbb{T}^d)$ such that $\hat{F}_\nu'\psi = \lambda\psi$. Let $\lambda = e^{i2\pi c}$ for some $c \in \mathbb{R}$, and set $\bar{\psi} = \frac{\psi}{h}$, where $h(x)$ is the density function of μ w.r.t. dx . By the definition of \hat{F}_ν' in (3.3) and $\mathcal{A}(y)$ in (3.4), we have

$$(3.7) \quad \sum_{Ty=x} \mathcal{A}(y) e^{i2\pi[\nu \cdot \tau(y) - c]} \bar{\psi}(y) = \bar{\psi}(x).$$

Now choose z such that $|\bar{\psi}(z)|$ obtains maximum. Since $\sum_{Ty=z} \mathcal{A}(y) = 1$, we must have $|\bar{\psi}(y)| = |\bar{\psi}(z)|$ for all $y \in T^{-1}(z)$. By induction, we get that $|\bar{\psi}(y)| = |\bar{\psi}(z)|$ for all $y \in \bigcup_{n=1}^\infty T^{-n}(z)$. Since T is mixing, the set $\bigcup_{n=1}^\infty T^{-n}(z)$ is dense in \mathbb{T}^d , and hence $|\bar{\psi}(x)| = |\bar{\psi}(z)|$ is constant. Thus (3.7) is a convex combination of points of a circle which lies on the circle. From this we deduce that

$$e^{i2\pi[\nu \cdot \tau(y) - c]} \bar{\psi}(y) = \bar{\psi}(Ty)$$

for all $y \in \mathbb{T}^d$, and hence (adjust c by an integer value if needed),

$$\nu \cdot \tau(y) = c - \frac{1}{2\pi} \arg \bar{\psi}(y) + \frac{1}{2\pi} \arg \bar{\psi}(Ty).$$

Since τ is not an essential coboundary over T , we must have $\nu = \mathbf{0}$. By integrating the last equation w.r.t. $d\mu$, we also have that $c = 0$ and thus $\lambda = 1$. Further, $\arg \bar{\psi} \equiv \text{constant}$ since it is T -invariant, and thus $\bar{\psi} \equiv \text{constant}$, which implies that $\psi = h\bar{\psi}$ is a constant multiple of h . Therefore, the space $\{\psi : \hat{F}_\nu'\psi = \psi\}$ is one-dimensional, so is the space $\{\varphi : \hat{F}_\nu\varphi = \varphi\}$ by duality. Since $\hat{F}_\nu 1 = 1$, we must have that φ is a constant function. \square

Now we are ready to prove the spectral gap property for $\hat{F} : \mathcal{W}^s \rightarrow \mathcal{W}^s$.

Proof of Theorem 2. To sum up, by Lemma 3.4 and 3.5, we have the following:

- (i) When $\nu = \mathbf{0}$, the spectrum $\text{Spec}(\hat{F}_\mathbf{0}) = \{1\} \cup \mathcal{K}_\mathbf{0}$, where 1 is a simple eigenvalue of $\hat{F}_\mathbf{0}$ and $\mathcal{K}_\mathbf{0}$ is a compact subset of the open unit disk \mathbb{D} . Equivalently, there are $M_\mathbf{0} > 0$, $r_\mathbf{0} \in [0, 1)$, and an orthogonal decomposition

$$H_\mathbf{0}^s(\mathbb{T}^d) = V_{\text{Const}} \oplus V_\mathbf{0},$$

such that $\|\hat{F}_\mathbf{0}^n|V_\mathbf{0}\| \leq M_\mathbf{0}r_\mathbf{0}^n$ for all $n \in \mathbb{N}$, where V_{Const} is the subspace consisting of constant functions;

- (ii) For all $\nu \in \mathbb{Z}^\ell \setminus \{\mathbf{0}\}$, the spectrum $\text{Spec}(\hat{F}_\nu)$ is strictly inside the open unit disk \mathbb{D} . Equivalently, there are $M_\nu > 0$ and $r_\nu \in [0, 1)$ such that $\|\hat{F}_\nu^n|H_\nu^s(\mathbb{T}^d)\| \leq M_\nu r_\nu^n$ for all $n \in \mathbb{N}$.

And by Proposition 3.2,

- (iii) When $|\nu| \geq \nu_1$, $\|\hat{F}_\nu^n|H_\nu^s(\mathbb{T}^d)\| \leq C_2\rho_2^n$ for all $n \in \mathbb{N}$.

We set the orthogonal direct sum

$$(3.8) \quad V = V_0 \oplus \left(\bigoplus_{\nu \in \mathbb{Z}^\ell \setminus \{0\}} H_\nu^s(\mathbb{T}^d) \right),$$

then $\mathcal{W}^s = V_{\text{Const}} \oplus V$. Furthermore, let $C_4 := \max\{C_2, \max_{|\nu| < \nu_1} \{M_\nu\}\}$ and $\rho_4 := \max\{\rho_2, \max_{|\nu| < \nu_1} \{r_\nu\}\}$, then we have $\|\hat{F}^n|V\| \leq C_4 \rho_4^n$ for all $n \in \mathbb{N}$. In other words, $\hat{F} = \bigoplus_{\nu \in \mathbb{Z}^\ell} \hat{F}_\nu$ has spectrum

$$\text{Spec}(\hat{F}) = \{1\} \cup \mathcal{K},$$

where $\mathcal{K} = \text{Spec}(\hat{F}|V) \subset \{z \in \mathbb{C} : |z| \leq \rho_4\}$, and 1 is the only leading simple eigenvalue with eigenvectors being constant functions. So $\hat{F} : \mathcal{W}^s \rightarrow \mathcal{W}^s$ has spectral gap. \square

3.4. Proof of Theorem 1. Now we use Theorem 2 to prove Theorem 1. What we need to do is to show that if $\hat{F} : \mathcal{W}^s \rightarrow \mathcal{W}^s$ has a spectral gap, then it is exponentially mixing. In the proof we regard the Hölder continuous observables ϕ and ψ as elements in \mathcal{W}^s and $(\mathcal{W}^s)'$ respectively.

Proof of Theorem 1. Since $\hat{F} : \mathcal{W}^s \rightarrow \mathcal{W}^s$ has a spectral gap, we can write

$$\hat{F} = \hat{F}|V_{\text{Const}} + \hat{F}|V =: \mathcal{P} + \mathcal{N},$$

where V is defined in (3.8). From the proof of Theorem 2, we know that

- (a) \mathcal{P} is a 1-dimensional projection, i.e., $\mathcal{P}^2 = \mathcal{P}$;
- (b) \mathcal{N} is a bounded operator with spectral radius $\text{Sp}(\mathcal{N}) \leq \rho_4 < 1$. In fact, $\|\mathcal{N}^n\| \leq C_4 \rho_4^n$ for all $n \in \mathbb{N}$;
- (c) $\mathcal{P}\mathcal{N} = \mathcal{N}\mathcal{P} = 0$.

Furthermore, 1 is the greatest simple eigenvalue for \hat{F}_0 with eigenvector 1 as well as for \hat{F}_0' with eigenvector h , and therefore, we rewrite $\mathcal{P} = 1 \otimes h \in \mathcal{W}^s \otimes (\mathcal{W}^s)'$.

Suppose $\phi, \psi \in C^\alpha(\mathbb{T}^{d+\ell})$ are given. Pick $s \in [-\alpha, 0)$ and let $\mathcal{W}^s = H^s(\mathbb{T}^d) \otimes H^{-s}(\mathbb{T}^\ell)$. Then the dual space of \mathcal{W}^s is $(\mathcal{W}^s)' = H^{-s}(\mathbb{T}^d) \otimes H^s(\mathbb{T}^\ell)$. Note that $C^\alpha(\mathbb{T}^{d+\ell})$ is contained in both \mathcal{W}^s and $(\mathcal{W}^s)'$, and thus $\phi \in \mathcal{W}^s$ and $\psi h \in (\mathcal{W}^s)'$, where $h \in C^\infty(\mathbb{T}^d)$ is the density function of μ w.r.t. dx . Hence,

$$\begin{aligned} \int (\phi \circ F^n) \psi dA &= \int (\phi \circ F^n) \psi h \, dx dy \\ &= (\psi h, \hat{F}^n(\phi))_{(\mathcal{W}^s)', \mathcal{W}^s} \\ &= (\psi h, \mathcal{P}(\phi))_{(\mathcal{W}^s)', \mathcal{W}^s} + (\psi h, \mathcal{N}^n(\phi))_{(\mathcal{W}^s)', \mathcal{W}^s} \\ &= (\psi h, (1 \otimes h)(\phi))_{(\mathcal{W}^s)', \mathcal{W}^s} + (\psi h, \mathcal{N}^n(\phi))_{(\mathcal{W}^s)', \mathcal{W}^s} \\ &= \left(\psi h, \left(\int \phi h dx dy \right) \cdot 1 \right)_{(\mathcal{W}^s)', \mathcal{W}^s} + (\psi h, \mathcal{N}^n(\phi))_{(\mathcal{W}^s)', \mathcal{W}^s} \\ &= \int \psi dA \int \phi dA + (\psi h, \mathcal{N}^n(\phi))_{(\mathcal{W}^s)', \mathcal{W}^s}. \end{aligned}$$

That is, the correlation function

$$C_n(\phi, \psi; F, dA) = |(\psi h, \mathcal{N}^n(\phi))_{(\mathcal{W}^s)', \mathcal{W}^s}| \leq \|\mathcal{N}^n\| \|\psi h\|_{(\mathcal{W}^s)'} \|\phi\|_{\mathcal{W}^s} \leq C_{\phi, \psi} \rho_4^n$$

where $C_{\phi, \psi} = C_4 \|\psi h\|_{(\mathcal{W}^s)'} \|\phi\|_{\mathcal{W}^s}$. \square

Remark 3.6. Using some Sobolev inequalities, it is not hard to get that $\|\psi h\|_{(\mathcal{W}^s)'} \leq C_5 \|\psi\|_{C^\alpha} \|h\|_{C^1}$ and $\|\phi\|_{\mathcal{W}^s} \leq C_6 \|\phi\|_{C^\alpha}$, and hence $C_{\phi, \psi} \leq C_7 \|\phi\|_{C^\alpha} \|\psi\|_{C^\alpha}$.

3.5. Proof of Theorem 3. Now we show the characters of the non-mixing skew products F_τ , that is, $\tau_1, \tau_2, \dots, \tau_\ell$ are integrally dependent mod \mathfrak{B} .

Proof of Theorem 3. (i) \Rightarrow (ii). Suppose $\tau_1, \tau_2, \dots, \tau_\ell$ are integrally dependent mod \mathfrak{B} , that is, there are $v \in \mathbb{Z}^\ell \setminus \{0\}$, $c \in \mathbb{R}$ and $u \in C^\infty(\mathbb{T}^d, \mathbb{R})$ such that

$$v \cdot \tau(x) = c + u(x) - u(Tx).$$

For any $(x, y) \in \mathbb{T}^d \times \mathbb{T}^\ell$, the set

$$\mathcal{L}(x, y) = \{(x', y') \in \mathbb{T}^d \times \mathbb{T}^\ell : v \cdot y' + u(x') = v \cdot y + u(x) \pmod{\mathbb{Z}}\}.$$

is well-defined. Moreover, since u is a smooth map, $\mathcal{L}(x, y)$ is a smooth $(d + \ell - 1)$ -dimensional manifold, and $\{\mathcal{L}(x, y) : (x, y) \in \mathbb{T}^d \times \mathbb{T}^\ell\}$ form a foliation of $\mathbb{T}^d \times \mathbb{T}^\ell$. It is clear that for any fixed $x \in \mathbb{T}^d$,

$$\mathcal{L}(x, y)|_{\{x\} \times \mathbb{T}^\ell} = \{(x, y') \in \{x\} \times \mathbb{T}^\ell : v \cdot (y - y') = 0 \pmod{\mathbb{Z}}\}.$$

It implies that the leaves of $\mathcal{L}(x, y)|_{\{x\} \times \mathbb{T}^\ell}$ are normal to v .

For $(x', y') \in \mathcal{L}(x, y)$, the definition of F gives

$$F(x, y) = (Tx, y + \tau(x)) \quad \text{and} \quad F(x', y') = (Tx', y' + \tau(x')),$$

then

$$v \cdot (y + \tau(x)) + u(Tx) = v \cdot y + v \cdot \tau(x) + u(Tx) = v \cdot y + c + u(x) \pmod{\mathbb{Z}}$$

and similarly $v \cdot (y' + \tau(x')) + u(Tx') = v \cdot y' + c + u(x') \pmod{\mathbb{Z}}$. Hence we obtain

$$v \cdot (y' + \tau(x')) + u(Tx') = v \cdot (y + \tau(x)) + u(Tx) \pmod{\mathbb{Z}}.$$

By definition of \mathcal{L} , we get $F(x', y') \in \mathcal{L}(F(x, y))$, that is, the foliation is F -invariant.

(ii) \Rightarrow (iii). Let p be a fixed point of the base map $T : \mathbb{T}^d \rightarrow \mathbb{T}^d$. Restricted to $\{p\} \times \mathbb{T}^\ell$ the leaves of the foliation \mathcal{L} become $(\ell - 1)$ -dimensional tori because the leaves are normal to v . Hence the quotient space $\{p\} \times \mathbb{T}^\ell / \sim$ is homeomorphic to a circle \mathbb{T} , where $(p, y) \sim (p, y')$ if (p, y) and (p, y') are in the same leave of $\mathcal{L}|_{\{p\} \times \mathbb{T}^\ell}$. Furthermore, let $\pi_p : \{p\} \times \mathbb{T}^\ell \rightarrow \mathbb{T}$ be the quotient map given by $\pi_p(p, y) = v \cdot y \pmod{\mathbb{Z}}$. Since $F|_{\{p\} \times \mathbb{T}^\ell} : \{p\} \times \mathbb{T}^\ell \rightarrow \{p\} \times \mathbb{T}^\ell$ is an fiberwise rotation given by $F(p, y) = (p, y + \tau(p) \pmod{\mathbb{Z}^\ell})$ and preserves the leaves, it induces a circle rotation $G_p : \mathbb{T} \rightarrow \mathbb{T}$ such that $\pi_p \circ F|_{\{p\} \times \mathbb{T}^\ell} = G_p \circ \pi_p$. It is easy to check that the rotation angle of G_p is given by $c = \pi_p(p, \tau(p)) \in \mathbb{T}$, and thus we also denote $G_p = R_c$.

π_p and G_p can be extended to maps $\pi : \mathbb{T}^d \times \mathbb{T}^\ell \rightarrow \mathbb{T}^d \times \mathbb{T}$ and $G : \mathbb{T}^d \times \mathbb{T} \rightarrow \mathbb{T}^d \times \mathbb{T}$ in a natural way. That is, for any $(x, y) \in \mathbb{T}^d \times \mathbb{T}^\ell$, let $(p, y') \in \mathcal{L}(x, y) \cap (\{p\} \times \mathbb{T}^\ell)$, and define $\pi(x, y) = (x, \pi_p(p, y'))$; and for any $(x, \bar{y}) \in \mathbb{T}^d \times \mathbb{T}$, define $G(x, \bar{y}) = (Tx, G_p(\bar{y})) = (Tx, R_c(\bar{y}))$. It is then easy to check that $\pi \circ F = G \circ \pi$.

(iii) \Rightarrow (iv). Weak mixing property does not hold for the circle rotation R_c , let alone the extension F .

(iv) \Rightarrow (i). It follows from the results by Parry and Pollicott [21], and also by Field and Parry [16]. \square

4. SPECTRAL PROPERTIES OF \hat{F}_ν : PROOF OF PROPOSITION 3.1 AND 3.2

We shall use the semiclassical analysis to prove Proposition 3.1 and Proposition 3.2. The flexibility of the parameter \hbar allows us to deal with the operators $\hat{F}_\nu : H_\nu^s(\mathbb{T}^d) \rightarrow H_\nu^s(\mathbb{T}^d)$, $\nu \in \mathbb{Z}^d$, in two different ways. To be precise, for any fixed frequency $\nu \in \mathbb{Z}^\ell$, we take $\hbar = 1$ and treat \hat{F}_ν as a classical FIO (up to a smoothing operator) in the proof of Proposition 3.1; while for Proposition 3.2, we set $\hbar = 1/\max\{1, |\nu|\}$ and regard \hat{F}_ν as an \hbar -scaled FIO (up to an \hbar -scaled smoothing operator).

4.1. The Sobolev spaces with non-standard inner products. We first construct a particular symbol on $T^*\mathbb{T}^d$ as follows. Choose

$$(4.1) \quad R > \max \left\{ 1, \frac{\max\{1, 2\|D\tau\|\}}{\gamma - 1} \right\},$$

where γ is given in (1.1), and $\|D\tau\| = \sup_{x \in \mathbb{T}^d} |D_x \tau|$. Let $g_0 \in C^\infty(\mathbb{R}^+)$ be such that

$$(4.2) \quad g_0(t) = \begin{cases} 1, & t \leq R; \\ t, & t \geq \frac{\gamma+1}{2}R, \end{cases}$$

and for $t \in [R, \frac{\gamma+1}{2}R)$, $g_0(t)$ is strictly increasing and $1 \leq g_0(t) \leq t$. Set $g(\xi) = g_0(|\xi|)$ for $\xi \in \mathbb{R}^d$. Given $s < 0$, define a symbol

$$(4.3) \quad \lambda_s(x, \xi) = h(x)^{\frac{1}{2}} g(\xi)^s \in S^s,$$

where $h(x)$ is the density function of μ w.r.t. dx . Further, given $\nu \in \mathbb{Z}^\ell$, define

$$(4.4) \quad \lambda_{s,\nu}(x, \xi) = \lambda_s \left(x, \frac{\xi}{\llbracket \nu \rrbracket} \right) \in S^s,$$

where $\llbracket \nu \rrbracket := \max\{1, |\nu|\}$.

Denote $\Lambda_{s,\nu} = \text{Op}(\lambda_{s,\nu}) \in \text{OPS}^s$, and define an inner product on $H^s(\mathbb{T}^d)$ by

$$\langle \varphi, \psi \rangle_{\Lambda_{s,\nu}} = \langle \Lambda_{s,\nu} \varphi, \Lambda_{s,\nu} \psi \rangle_{L^2}, \quad \varphi, \psi \in H^s(\mathbb{T}^d).$$

When equipped with $\langle \cdot, \cdot \rangle_{\Lambda_{s,\nu}}$, $H^s(\mathbb{T}^d)$ is denoted by $H_{\Lambda_{s,\nu}}(\mathbb{T}^d)$ instead. The Sobolev space $H_{\Lambda_{s,\nu}}$ is unitarily equivalent to the L^2 space, that is,

$$\Lambda_{s,\nu} H_{\Lambda_{s,\nu}}(\mathbb{T}^d) \cong L^2(\mathbb{T}^d), \quad \text{or} \quad H_{\Lambda_{s,\nu}}(\mathbb{T}^d) \cong \Lambda_{s,\nu}^{-1} L^2(\mathbb{T}^d).$$

We claim that the spaces $H_{\Lambda_{s,\nu}}(\mathbb{T}^d)$ and $H_\nu^s(\mathbb{T}^d)$, which are identical as the set of s -order Sobolev functions, have comparable inner products in the following sense: there is $C_1 = C_1(d, s) > 0$ such that

$$(4.5) \quad \frac{1}{C_1} |\langle \varphi, \psi \rangle_{s,\nu}| \leq |\langle \varphi, \psi \rangle_{\Lambda_{s,\nu}}| \leq C_1 |\langle \varphi, \psi \rangle_{s,\nu}|, \quad \text{for any } \varphi, \psi \in H^s(\mathbb{T}^d).$$

To see this, recall that $H_\nu^s(\mathbb{T}^d)$ is equipped with the $\langle \nu \rangle^{-1}$ -scaled s -inner product $\langle \cdot, \cdot \rangle_{s,\nu}$ given by (3.1). Alternatively, we have

$$\text{Op}(\langle \nu \rangle^{-s} \langle \xi \rangle^s) H_\nu^s(\mathbb{T}^d) \cong L^2(\mathbb{T}^d).$$

Then the comparability of inner products simply follows from that $\lambda_{s,\nu}(x, \xi) = \langle \nu \rangle^{-s} \langle \xi \rangle^s$, i.e., there is $C_0 = C_0(d, s) > 0$ such that for any $\nu \in \mathbb{Z}^\ell$,

$$\frac{1}{C_0} \leq \frac{\lambda_{s,\nu}(x, \xi)}{\langle \nu \rangle^{-s} \langle \xi \rangle^s} \leq C_0, \quad \text{for any } (x, \xi) \in T^*\mathbb{T}^d.$$

4.2. Proof of Proposition 3.1. Recall that $\hat{F}_\nu : H_\nu^s(\mathbb{T}^d) \rightarrow H_\nu^s(\mathbb{T}^d)$ is defined in (3.2). Switching to the inner product $\langle \cdot, \cdot \rangle_{\Lambda_{s,\nu}}$, we mainly study the operator $\hat{F}_\nu : H_{\Lambda_{s,\nu}}(\mathbb{T}^d) \rightarrow H_{\Lambda_{s,\nu}}(\mathbb{T}^d)$ instead.

Proof of Proposition 3.1. Let $s < 0$ and $\nu \in \mathbb{Z}^\ell$ be fixed. By the formula of \hat{F}_ν in (3.2), the Fourier transform (2.2) and inverse transform (2.3), we rewrite

$$\hat{F}_\nu \varphi(x) = \sum_{\xi \in \mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i2\pi\nu \cdot \tau(x)} e^{i2\pi[Tx \cdot \xi - y \cdot \xi]} \varphi(y) dy.$$

As discussed in Subsection 2.2.7, \hat{F}_ν is regarded as a classical (i.e., $\hbar = 1$) toroidal FSO with amplitude $a_{\text{tor}}^\nu = a^\nu|_{\mathbb{T}^d \times \mathbb{Z}^d}$ and phase $S_{\text{tor}} = S|_{\mathbb{T}^d \times \mathbb{Z}^d}$, where $a^\nu(x, \xi) = e^{i2\pi\nu \cdot \tau(x)} \in S^0$ and $S(x, \xi) = Tx \cdot \xi \in S^1$. Moreover,

$$(4.6) \quad \hat{F}_\nu = \tilde{F}_\nu + \tilde{K}_\nu,$$

where $\tilde{F}_\nu = \Phi(a^\nu, S)$ is the classical (periodic Euclidean) FIO with amplitude a^ν and phase S (see Definition 2.7 with $\hbar = 1$), and \tilde{K}_ν is a smoothing operator. It is thus sufficient to study the spectral property of \tilde{F}_ν . Note that the canonical transformation $\mathcal{F} : (x, \xi) \mapsto (y, \eta)$ associated to \tilde{F}_ν is given by

$$y = Tx, \quad \eta = [(D_x T)^t]^{-1} \xi,$$

which is irrelevant to ν since the phase function S is independent of ν .

We have the following commutative diagram

$$\begin{array}{ccc} H_{\Lambda_{s,\nu}}(\mathbb{T}^d) & \xrightarrow{\tilde{F}_\nu} & H_{\Lambda_{s,\nu}}(\mathbb{T}^d) \\ \Lambda_{s,\nu} \downarrow & & \downarrow \Lambda_{s,\nu} \\ L^2(\mathbb{T}^d) & \xrightarrow{Q_\nu} & L^2(\mathbb{T}^d) \end{array}$$

where $Q_\nu = \Lambda_{s,\nu} \tilde{F}_\nu \Lambda_{s,\nu}^{-1}$. We then set

$$\begin{aligned} P_\nu &= Q_\nu^* Q_\nu = (\Lambda_{s,\nu}^{-1})^* \left[\tilde{F}_\nu^* (\Lambda_{s,\nu}^* \Lambda_{s,\nu}) \tilde{F}_\nu \right] \Lambda_{s,\nu}^{-1} \\ &= (\text{Op}(\lambda_{s,\nu})^{-1})^* \{ \Phi(a^\nu, S)^* [\text{Op}(\lambda_{s,\nu})^* \text{Op}(\lambda_{s,\nu})] \Phi(a^\nu, S) \} \text{Op}(\lambda_{s,\nu})^{-1}. \end{aligned}$$

By the symbol calculus (Theorem 2.10) and the Egorov's theorem (Theorem 2.15), the operator P_ν is a classical PDO of order 0. Denote by $p_\nu(x, \xi)$ the symbol of P_ν . By the L^2 -continuity theorem (Theorem 2.18), $P_\nu : L^2(\mathbb{T}^d) \rightarrow L^2(\mathbb{T}^d)$ is a bounded operator such that for any $\varepsilon > 0$, we can write $P_\nu = K_\nu^0(\varepsilon) + R_\nu^0(\varepsilon) =: K_\nu^0 + R_\nu^0$, where K_ν^0 is compact; and moreover, by Lemma 4.1 below, the definition of g in Section 4.1 and the definition γ in (1.1), we get

$$\begin{aligned} \|R_\nu^0\|_{L^2 \rightarrow L^2} &\leq \sup_x \limsup_{|\xi| \rightarrow \infty} |p_\nu(x, \xi)| + \varepsilon \\ &= \sup_x \limsup_{|\xi| \rightarrow \infty} \sum_{x=Ty} \mathcal{A}(y) \left(\frac{g((D_y T)^t(\xi/\|\nu\|))}{g(\xi/\|\nu\|)} \right)^{2s} + \varepsilon \\ &\leq \sup_x \sum_{x=Ty} \mathcal{A}(y) \limsup_{|\xi| \rightarrow \infty} \left(\frac{|(D_y T)^t \xi|}{|\xi|} \right)^{2s} + \varepsilon \\ &\leq \sup_x \sum_{x=Ty} \mathcal{A}(y) \gamma^{2s} + \varepsilon = \gamma^{2s} + \varepsilon. \end{aligned}$$

Choose $\varepsilon > 0$ small enough such that

$$(4.7) \quad \rho_1 := \sqrt{\gamma^{2s} + \varepsilon} < 1.$$

By the polar decomposition, $Q_\nu = \sqrt{P_\nu} U_\nu$ for some partial isometry $U_\nu : L^2(\mathbb{T}^d) \rightarrow L^2(\mathbb{T}^d)$, and thus there is also a decomposition $Q_\nu = K_\nu^1 + R_\nu^1$ such that K_ν^1 is compact and $\|R_\nu^1\|_{L^2 \rightarrow L^2} \leq \rho_1$. By unitary equivalence between $Q_\nu : L^2(\mathbb{T}^d) \rightarrow L^2(\mathbb{T}^d)$ and $\tilde{F}_\nu : H_{\Lambda_{s,\nu}}(\mathbb{T}^d) \rightarrow H_{\Lambda_{s,\nu}}(\mathbb{T}^d)$, there is a similar decomposition $\tilde{F}_\nu = K_\nu^2 + R_\nu$ such that K_ν^2 is compact, and $\|R_\nu|_{H_{\Lambda_{s,\nu}}(\mathbb{T}^d)}\| \leq \rho_1$. By (4.6), we have

$$(4.8) \quad \hat{F}_\nu = \tilde{F}_\nu + \tilde{K}_\nu = (K_\nu^2 + \tilde{K}_\nu) + R_\nu =: K_\nu + R_\nu,$$

and note that $K_\nu = K_\nu^2 + \tilde{K}_\nu$ is a compact operator.

Switching back to the inner product $\langle \cdot, \cdot \rangle_{s,\nu}$, we have that $\hat{F}_\nu : H_\nu^s(\mathbb{T}^d) \rightarrow H_\nu^s(\mathbb{T}^d)$ takes exactly the same decomposition (4.8), and K_ν is still compact. By the choice of the constant C_1 in (4.5), we get

$$\|R_\nu^n|_{H_\nu^s(\mathbb{T}^d)}\| \leq C_1 \|R_\nu^n|_{H_{\Lambda_{s,\nu}}(\mathbb{T}^d)}\| \leq C_1 \|R_\nu|_{H_{\Lambda_{s,\nu}}(\mathbb{T}^d)}\|^n \leq C_1 \rho_1^n.$$

This completes the proof of Proposition 3.1. \square

Lemma 4.1. $P_\nu \in \text{OPS}^0$ has a symbol

$$p_\nu(x, \xi) = \sum_{y=Tx} \mathcal{A}(y) \left(\frac{g((D_y T)^t(\xi/\llbracket \nu \rrbracket))}{g(\xi/\llbracket \nu \rrbracket)} \right)^{2s} \pmod{S^{-1}},$$

where $\mathcal{A}(y)$ is defined in (3.4).

Proof. Note that $\Lambda_{s,\nu} \in \text{OPS}^s$ has a symbol $\lambda_{s,\nu}$ given by (4.4). By Theorem 2.10, $\Lambda_{s,\nu}^* \in \text{OPS}^s$ has a symbol $\lambda_{s,\nu} \pmod{S^{s-1}}$; $\Lambda_{s,\nu}^{-1} \in \text{OPS}^{-s}$ has a symbol $\lambda_{s,\nu}^{-1} \pmod{S^{-s-1}}$, and so does $(\Lambda_{s,\nu}^{-1})^* \in \text{OPS}^{-s}$. Further, $\Lambda_{s,\nu}^* \Lambda_{s,\nu} \in \text{OPS}^{2s}$ has a symbol $\lambda_{s,\nu}^2 \pmod{S^{2s-1}}$. Then by Egorov's theorem 2.15, $\tilde{F}_\nu^*(\Lambda_{s,\nu}^* \Lambda_{s,\nu}) \tilde{F}_\nu \in \text{OPS}^{2s}$ has a symbol

$$\begin{aligned} \tilde{a}(y, \eta) &= \sum_{\substack{y=Tx, \\ \eta=[(D_x T)^t]^{-1}\xi}} \lambda_{s,\nu}^2(x, \xi) \left| e^{i2\pi\nu \cdot \tau(x)} \right|^2 |\det(D_x T)^t|^{-1} \pmod{S^{2s-1}} \\ &= \sum_{y=Tx} \frac{\lambda_{s,\nu}^2(x, (D_x T)^t \eta)}{|\text{Jac}(T)(x)|} \pmod{S^{2s-1}}. \end{aligned}$$

Use the composition rule and recall the definition of $\lambda_{s,\nu}$ in (4.4), we have $P_\nu \in \text{OPS}^0$ with a symbol

$$\begin{aligned} p_\nu(y, \eta) &= \sum_{y=Tx} \frac{\lambda_{s,\nu}^2(x, (D_x T)^t \eta)}{|\text{Jac}(T)(x)|} \frac{1}{\lambda_{s,\nu}^2(y, \eta)} \pmod{S^{-1}} \\ &= \sum_{y=Tx} \frac{1}{|\text{Jac}(T)(x)|} \frac{h(x)}{h(y)} \frac{g((D_x T)^t(\eta/\llbracket \nu \rrbracket))^{2s}}{g(\eta/\llbracket \nu \rrbracket)^{2s}} \pmod{S^{-1}} \\ &= \sum_{y=Tx} \mathcal{A}(x) \left(\frac{g((D_x T)^t(\eta/\llbracket \nu \rrbracket))}{g(\eta/\llbracket \nu \rrbracket)} \right)^{2s} \pmod{S^{-1}}. \end{aligned}$$

This is what we need. \square

4.3. Proof of Proposition 3.2. To prove Proposition 3.2, we relate the semiclassical parameter \hbar with a given frequency $\boldsymbol{\nu} \in \mathbb{Z}^\ell$ by setting $\hbar = 1/\llbracket \boldsymbol{\nu} \rrbracket$. In this way, we study the operator $\hat{F}_{\boldsymbol{\nu}}^n$ as an \hbar -scaled toroidal FSO, and hence separate the dependence of $\hat{F}_{\boldsymbol{\nu}}^n$ on the frequency $\boldsymbol{\nu}$ into two parts: the dependence on the modulus $\llbracket \boldsymbol{\nu} \rrbracket = 1/\hbar$ and that on the direction vector $\mathbf{n}_{\boldsymbol{\nu}} := \boldsymbol{\nu}/\llbracket \boldsymbol{\nu} \rrbracket$. The key step of the proof is the estimate stated and proved in Lemma 5.1 in the next section.

Proof of Proposition 3.2. Let $s < 0$ and assume that τ is not an essential coboundary over $T : \mathbb{T}^d \rightarrow \mathbb{T}^d$. Given $\boldsymbol{\nu} \in \mathbb{Z}^\ell$, set $\hbar = 1/\llbracket \boldsymbol{\nu} \rrbracket$, where $\llbracket \boldsymbol{\nu} \rrbracket = \max\{1, |\boldsymbol{\nu}|\}$. Also denote by $\mathbf{n}_{\boldsymbol{\nu}} = \boldsymbol{\nu}/\llbracket \boldsymbol{\nu} \rrbracket$ the direction vector of $\boldsymbol{\nu}$, and note that either $\mathbf{n}_{\boldsymbol{\nu}} = \mathbf{0}$ (only if $\boldsymbol{\nu} = \mathbf{0}$) or $\mathbf{n}_{\boldsymbol{\nu}}$ lies on the $(\ell - 1)$ -dimensional unit sphere $\mathbb{S}^{\ell-1}$.

For any $n \in \mathbb{N}$, the operator $\hat{F}_{\boldsymbol{\nu}}^n$ can then be rewritten as

$$\begin{aligned} \hat{F}_{\boldsymbol{\nu}}^n \varphi(x) &= \varphi(T^n x) e^{i2\pi \boldsymbol{\nu} \cdot \sum_{k=0}^{n-1} \tau(T^k x)} \\ &= \sum_{\xi \in \mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i2\pi \{ [T^n x \cdot \xi + \boldsymbol{\nu} \cdot \sum_{k=0}^{n-1} \tau(T^k x)] - y \cdot \xi \}} \varphi(y) dy \\ &= \sum_{\xi \in \hbar \mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i2\pi \frac{1}{\hbar} \{ [T^n x \cdot \xi + \mathbf{n}_{\boldsymbol{\nu}} \cdot \sum_{k=0}^{n-1} \tau(T^k x)] - y \cdot \xi \}} \varphi(y) dy. \end{aligned}$$

That is, $\hat{F}_{\boldsymbol{\nu}}^n$ is regarded as an \hbar -scaled toroidal FSO with amplitude $a_{\text{tor}} \equiv 1$ and phase $(S_{\mathbf{n}_{\boldsymbol{\nu}}, n})_{\text{tor}} = S_{\mathbf{n}_{\boldsymbol{\nu}}, n}|_{\mathbb{T}^d \times \mathbb{Z}^d}$, where

$$(4.9) \quad S_{\mathbf{n}, n}(x, \xi) = T^n x \cdot \xi + \mathbf{n} \cdot \sum_{k=0}^{n-1} \tau(T^k x) \quad \text{for any } \mathbf{n} \in \mathbb{R}^\ell, (x, \xi) \in T^* \mathbb{T}^d.$$

From what we have discussed in Section 2.2.7, we have that $\hat{F}_{\boldsymbol{\nu}}^n = \tilde{F}_{\boldsymbol{\nu}, n} + \tilde{K}_{\boldsymbol{\nu}, n}$, where $\tilde{F}_{\boldsymbol{\nu}, n}$ is the \hbar -scaled (periodic Euclidean) FIO with amplitude $a \equiv 1$ and phase $S_{\mathbf{n}_{\boldsymbol{\nu}}, n}$, and $\tilde{K}_{\boldsymbol{\nu}, n}$ is a smoothing operator. Moreover, by Lemma 4.2 below, there exists $L_1(n) > 0$ independent of $\boldsymbol{\nu}$ such that $\|\tilde{K}_{\boldsymbol{\nu}, n}|_{H_{\Lambda_{s, \boldsymbol{\nu}}}(\mathbb{T}^d)}\| \leq \hbar L_1(n)$.

We thus focus on the spectral properties of the \hbar -scaled FIO $\tilde{F}_{\boldsymbol{\nu}, n} = \Phi_{\hbar}(1, S_{\mathbf{n}_{\boldsymbol{\nu}}, n})$. Note that the canonical transformation $\mathcal{F}_{\hbar} : (x, \hbar \xi) \mapsto (y, \hbar \eta)$ associated to $\tilde{F}_{\boldsymbol{\nu}, n}$ is given by

$$y = T^n x, \quad \eta = [(D_x T^n)^t]^{-1} [\xi - W_{\mathbf{n}_{\boldsymbol{\nu}}, n}(x)],$$

where

$$(4.10) \quad W_{\mathbf{n}, n}(x) = W_n(x) \mathbf{n} \quad \text{and} \quad W_n(x) = \sum_{k=0}^{n-1} (D_x T^k)^t (D_{T^k x} \tau)^t.$$

By (4.3), (4.4) and that $\hbar = 1/\llbracket \boldsymbol{\nu} \rrbracket$, we rewrite $\lambda_{s, \boldsymbol{\nu}}(x, \xi) = \lambda_s(x, \hbar \xi)$, and hence $\Lambda_{s, \boldsymbol{\nu}} = \text{Op}(\lambda_{s, \boldsymbol{\nu}}) = \text{Op}_{\hbar}(\lambda_s) \in \text{Op}_{\hbar} S^s$. The following commutative diagram

$$\begin{array}{ccc} H_{\Lambda_{s, \boldsymbol{\nu}}}(\mathbb{T}^d) & \xrightarrow{\tilde{F}_{\boldsymbol{\nu}, n}} & H_{\Lambda_{s, \boldsymbol{\nu}}}(\mathbb{T}^d) \\ \Lambda_{s, \boldsymbol{\nu}} \downarrow & & \downarrow \Lambda_{s, \boldsymbol{\nu}} \\ L^2(\mathbb{T}^d) & \xrightarrow{\tilde{Q}_{\boldsymbol{\nu}, n}} & L^2(\mathbb{T}^d) \end{array}$$

suggests that we should instead study the operator

$$(4.11) \quad \begin{aligned} \tilde{P}_{\nu,n} &= \tilde{Q}_{\nu,n}^* \tilde{Q}_{\nu,n} = (\Lambda_{s,\nu}^{-1})^* \left[\tilde{F}_{\nu,n}^* (\Lambda_{s,\nu}^* \Lambda_{s,\nu}) \tilde{F}_{\nu,n} \right] \Lambda_{s,\nu}^{-1} \\ &= (\text{Op}_{\hbar}(\lambda_s)^{-1})^* \{ (\Phi_{\hbar}(1, S_{\mathbf{n}_{\nu},n}))^* [\text{Op}_{\hbar}(\lambda_s)^* \text{Op}_{\hbar}(\lambda_s)] \Phi_{\hbar}(1, S_{\mathbf{n}_{\nu},n}) \} \text{Op}_{\hbar}(\lambda_s)^{-1}. \end{aligned}$$

By the \hbar -scaled symbol calculus (Theorem 2.12) and the \hbar -scaled version of Egorov's theorem (Theorem 2.15), we have that $\tilde{P}_{\nu,n} \in \text{Op}_{\hbar} S^0$, and it has a symbol of the form $(\tilde{p}_{\mathbf{n}_{\nu},n} + \hbar \tilde{r}_{\mathbf{n}_{\nu},n})$ given by Lemma 4.3 below. Moreover, by the \hbar -scaled L^2 -continuity theorem (Theorem 2.19) and part (2) in Lemma 4.3, we get that there exists $L_2(n) > 0$ independent of ν such that $C_{k_2}(\tilde{p}_{\mathbf{n}_{\nu},n}, \tilde{r}_{\mathbf{n}_{\nu},n}) \leq L_2(n)$, then

$$(4.12) \quad \begin{aligned} \|\tilde{P}_{\nu,n}|L^2(\mathbb{T}^d)\| &\leq \sup_{(x,\xi) \in T^*\mathbb{T}^d} |\tilde{p}_{\mathbf{n}_{\nu},n}(x,\xi)| + \hbar C_{k_2}(\tilde{p}_{\mathbf{n}_{\nu},n}, \tilde{r}_{\mathbf{n}_{\nu},n}) \\ &\leq \sup_{(x,\xi) \in T^*\mathbb{T}^d} \tilde{p}_{\mathbf{n}_{\nu},n}(x,\xi) + \hbar L_2(n). \end{aligned}$$

By Sublemma 5.2 (1) in the next section, we have that for any $n \in \mathbb{N}$,

$$\|\tilde{P}_{\nu,n}|L^2(\mathbb{T}^d)\| \leq 1 + L_2(n),$$

and hence

$$(4.13) \quad \begin{aligned} \|\hat{F}_{\nu}^n|H_{\Lambda_{s,\nu}}(\mathbb{T}^d)\| &\leq \|\tilde{F}_{\nu,n}|H_{\Lambda_{s,\nu}}(\mathbb{T}^d)\| + \|\tilde{K}_{\nu,n}|H_{\Lambda_{s,\nu}}(\mathbb{T}^d)\| \\ &\leq \sqrt{\|\tilde{P}_{\nu,n}|L^2(\mathbb{T}^d)\|} + \hbar L_1(n) \\ &\leq \sqrt{1 + L_2(n)} + L_1(n), \end{aligned}$$

where $L_1(n)$ is given by Lemma 4.2.

Furthermore, by (4.12) and our key lemma - Lemma 5.1 in the next section, there are $n_0 \in \mathbb{N}$ and $\tilde{p}_0 < 1$ such that for all $\nu \in \mathbb{Z}^\ell \setminus \{\mathbf{0}\}$,

$$(4.14) \quad \|\tilde{P}_{\nu,n_0}|L^2(\mathbb{T}^d)\| \leq \sup_{(x,\xi) \in T^*\mathbb{T}^d} |\tilde{p}_{\mathbf{n}_{\nu},n_0}(x,\xi)| + \hbar L_2(n_0) \leq \tilde{p}_0 + \frac{L_2(n_0)}{\llbracket \nu \rrbracket}.$$

Choose $\nu_1 > 0$ such that

$$(4.15) \quad \rho_2 := \left(\sqrt{\tilde{p}_0 + \frac{L_2(n_0)}{\nu_1}} + \frac{L_1(n_0)}{\nu_1} \right)^{1/n_0} < 1.$$

By (4.14) and Lemma 4.2, we have for all $\nu \in \mathbb{Z}^\ell \setminus \{\mathbf{0}\}$ with $\llbracket \nu \rrbracket = |\nu| \geq \nu_1$,

$$(4.16) \quad \begin{aligned} \|\hat{F}_{\nu}^{n_0}|H_{\Lambda_{s,\nu}}(\mathbb{T}^d)\| &\leq \|\tilde{F}_{\nu,n_0}|H_{\Lambda_{s,\nu}}(\mathbb{T}^d)\| + \|\tilde{K}_{\nu,n_0}|H_{\Lambda_{s,\nu}}(\mathbb{T}^d)\| \\ &\leq \sqrt{\|\tilde{P}_{\nu,n_0}|L^2(\mathbb{T}^d)\|} + \hbar L_1(n_0) \\ &\leq \sqrt{\tilde{p}_0 + \frac{L_2(n_0)}{\llbracket \nu \rrbracket}} + \frac{L_1(n_0)}{\llbracket \nu \rrbracket} \leq \rho_2^{n_0}. \end{aligned}$$

Now for any $n \in \mathbb{N}$, we write $n = kn_0 + j$, where $k \in \mathbb{N}_0$ and $0 \leq j < n_0$. Then by (4.13) and (4.16), we have for all $\nu \in \mathbb{Z}^\ell$ with $|\nu| \geq \nu_1$,

$$\begin{aligned} \|\hat{F}_{\nu}^n|H_{\Lambda_{s,\nu}}(\mathbb{T}^d)\| &\leq \|\hat{F}_{\nu}^{n_0}|H_{\Lambda_{s,\nu}}(\mathbb{T}^d)\|^k \|\hat{F}_{\nu}^j|H_{\Lambda_{s,\nu}}(\mathbb{T}^d)\| \\ &\leq \rho_2^{kn_0} \left[\sqrt{1 + L_2(j)} + L_1(j) \right] \leq C'_2 \rho_2^n, \end{aligned}$$

where we set

$$C'_2 := \max_{1 \leq j < n_0} \frac{\sqrt{1 + L_2(j)} + L_1(j)}{\rho_2^j}.$$

Switch back to the inner product $\langle \cdot, \cdot \rangle_{s, \nu}$, and recall the choice of C_1 in (4.5). We take $C_2 = C_1 C'_2$, then for all $\nu \in \mathbb{Z}^\ell$ with $|\nu| \geq \nu_1$,

$$\|\hat{F}_\nu^n |H_\nu^s(\mathbb{T}^d)\| \leq C_2 \rho_2^n.$$

This completes the proof of Proposition 3.2. \square

Lemma 4.2. *For any $n \in \mathbb{N}$, there is $L_1(n) = L_1(n; d, s, T, \tau)$, which is independent of $\nu \in \mathbb{Z}^\ell$, such that $\|\tilde{K}_{\nu, n} |H_{\Lambda_{s, \nu}}(\mathbb{T}^d)\| \leq \hbar L_1(n)$.*

Proof. By Section 2.2.7, $\tilde{K}_{\nu, n}$ is an \hbar -scaled smoothing operator that depends on the amplitude $a \equiv 1$ and phase $S_{\mathbf{n}, n}$, where $S_{\mathbf{n}, n}$ is given by (4.9). Moreover, $\|\tilde{K}_{\nu, n} |H_h^s(\mathbb{T}^d)\| \leq \hbar C_{k_3}(1, S_{\mathbf{n}, n})$, where $C_{k_3}(1, S_{\mathbf{n}, n})$ only depends on the seminorms $\mathcal{N}_{k_3}(1) = 1$ and

$$(4.17) \quad \begin{aligned} \mathcal{N}_{k_3+2}(S_{\mathbf{n}, n}) &\leq \mathcal{N}_{k_3+2}(T^n x \cdot \xi) + |\mathbf{n}_\nu| \mathcal{N}_{k_3+2} \left(\left| \sum_{k=0}^{n-1} \tau(T^k x) \right| \right) \\ &\leq \mathcal{N}_{k_3+2}(T^n x \cdot \xi) + \mathcal{N}_{k_3+2} \left(\left| \sum_{k=0}^{n-1} \tau(T^k x) \right| \right), \end{aligned}$$

for some $k_3 \in \mathbb{N}$ that only relies on d and s . Hence, there is $L'_1(n) = L'_1(n; d, s, T, \tau)$ that depends on n, d, s, T, τ but not on ν , such that $C_{k_3}(1, S_{\mathbf{n}, n}) \leq L'_1(n)$ and thus $\|\tilde{K}_{\nu, n} |H_h^s(\mathbb{T}^d)\| \leq \hbar L'_1(n)$.

Recall that $H_h^s(\mathbb{T}^d)$ is the space of s -order Sobolev functions endowed with the \hbar -scaled s -inner product, or alternatively, $\text{Op}(\hbar^s \langle \xi \rangle^s) H_h^s(\mathbb{T}^d) \cong L^2(\mathbb{T}^d)$. Also, $\Lambda_{s, \nu} H_{\Lambda_{s, \nu}}(\mathbb{T}^d) \cong L^2(\mathbb{T}^d)$, where $\Lambda_{s, \nu} = \text{Op}(\lambda_{s, \nu})$. Since $\lambda_{s, \nu}(x, \xi) = \lambda_s(x, \hbar \xi) \asymp \hbar^s \langle \xi \rangle^s$ for all $(x, \xi) \in T^* \mathbb{T}^d$, by similar reasons as for (4.5), there exists a constant $C'_1 = C'_1(d, s) > 0$ such that

$$\frac{1}{C'_1} |\langle \varphi, \psi \rangle_{s, \hbar}| \leq |\langle \varphi, \psi \rangle_{\Lambda_{s, \nu}}| \leq C'_1 |\langle \varphi, \psi \rangle_{s, \hbar}|, \quad \varphi, \psi \in H^s(\mathbb{T}^d).$$

Set $L_1(n) = C'_1 L'_1(n)$. We then have

$$\|\tilde{K}_{\nu, n} |H_{\Lambda_{s, \nu}}(\mathbb{T}^d)\| \leq C'_1 \|\tilde{K}_{\nu, n} |H_h^s(\mathbb{T}^d)\| \leq \hbar C'_1 L'_1(n) = \hbar L_1(n).$$

\square

Lemma 4.3. *Given $\nu \in \mathbb{Z}^\ell$, let $\hbar = 1/\|\nu\|$. For any $n \in \mathbb{N}$, $\tilde{P}_{\nu, n} \in \text{Op}_\hbar S^0$ has a symbol of the form $\tilde{p}_{\nu, n} + \hbar \tilde{r}_{\nu, n}$, where $\tilde{p}_{\nu, n} \in S^0$ and $\tilde{r}_{\nu, n} \in S^{-1}$, such that*

(1) $\tilde{p}_{\nu, n}$ is positive, and is given by

$$(4.18) \quad \tilde{p}_{\nu, n}(x, \xi) = \sum_{x=T^n y} \mathcal{A}_n(y) \left(\frac{g((D_y T^n)^t \xi + W_{\mathbf{n}, n}(y))}{g(\xi)} \right)^{2s},$$

where $\mathcal{A}_n(y)$ is given in (3.5) and $W_{\mathbf{n}, n}(y)$ is given in (4.10);

(2) Let $C_{k_2}(\cdot, \cdot)$ be as introduced in Theorem 2.19. There is $L_2(n) = L_2(n; d, T, \tau)$ independent of ν such that

$$C_{k_2}(\tilde{p}_{\nu, n}, \tilde{r}_{\nu, n}) \leq L_2(n).$$

Proof. Recall that $\Lambda_{s,\nu} = \text{Op}_{\hbar}(\lambda_s) \in \text{Op}_{\hbar} S^s$. By Theorem 2.12, $\Lambda_{s,\nu}^* \in \text{Op}_{\hbar} S^s$ has a symbol $\lambda_s \pmod{\hbar S^{s-1}}$, and $\Lambda_{s,\nu}^{-1}, (\Lambda_{s,\nu}^{-1})^* \in \text{Op}_{\hbar} S^{-s}$ both have a symbol $\lambda_s^{-1} \pmod{\hbar S^{-s-1}}$. Further, $\Lambda_{s,\nu}^* \Lambda_{s,\nu} \in \text{Op}_{\hbar} S^{2s}$ has a symbol $\lambda_s^2 \pmod{\hbar S^{2s-1}}$. By the \hbar -scaled version of the Egorov's theorem (see Theorem 2.15 and Remark 2.16), $\tilde{F}_{\nu,n}^*(\Lambda_{s,\nu}^* \Lambda_{s,\nu}) \tilde{F}_{\nu,n} \in \text{Op}_{\hbar} S^{2s}$ has a symbol

$$\begin{aligned} \tilde{a}_n(y, \eta) &= \sum_{\substack{y=T^n x, \\ \eta=[(D_x T^n)^t]^{-1}[\xi - W_{\mathbf{n}\nu,n}(x)]}} \lambda_s^2(x, \xi) \cdot 1^2 \cdot |\det(D_x T^n)^t|^{-1} \pmod{\hbar S^{2s-1}} \\ &= \sum_{y=T^n x} \frac{\lambda_s^2(x, (D_x T^n)^t \eta + W_{\mathbf{n}\nu,n}(x))}{|\text{Jac}(T^n)(x)|} \pmod{\hbar S^{2s-1}}. \end{aligned}$$

By composition rule again, $\tilde{P}_{\nu,n} \in \text{Op}_{\hbar} S^0$ has a symbol

$$\begin{aligned} \tilde{p}_{\mathbf{n}\nu,n}(y, \eta) &= \sum_{y=T^n x} \frac{\lambda_s^2(x, (D_x T^n)^t \eta + W_{\mathbf{n}\nu,n}(x))}{|\text{Jac}(T^n)(x)|} \frac{1}{\lambda_s^2(y, \eta)} \pmod{\hbar S^{-1}} \\ &= \sum_{y=T^n x} \frac{1}{|\text{Jac}(T^n)(x)|} \frac{h(x)}{h(y)} \frac{g((D_x T^n)^t \eta + W_{\mathbf{n}\nu,n}(x))^{2s}}{g(\eta)^{2s}} \pmod{\hbar S^{-1}} \\ &= \sum_{y=T^n x} \mathcal{A}_n(x) \left(\frac{g((D_x T^n)^t \eta + W_{\mathbf{n}\nu,n}(x))}{g(\eta)} \right)^{2s} \pmod{\hbar S^{-1}}. \end{aligned}$$

This finishes the proof of the first part.

Since all the above modulo terms are calculated from the symbol λ_s and the phase $S_{\mathbf{n}\nu,n}$, which depends on $\mathbf{n}\nu$ but not $[\nu]$, we can write the full symbol of $\tilde{P}_{\nu,n}$ by $\tilde{p}_{\mathbf{n}\nu,n} + \hbar \tilde{r}_{\mathbf{n}\nu,n}$ for some $\tilde{r}_{\mathbf{n}\nu,n} \in S^{-1}$.

By Theorem 2.19, $C_{k_2}(\tilde{p}_{\mathbf{n}\nu,n}, \tilde{r}_{\mathbf{n}\nu,n})$ is bounded by a constant which only depends on the \mathcal{N}_{k_2} -seminorms of $\tilde{p}_{\mathbf{n}\nu,n}$ and $\tilde{r}_{\mathbf{n}\nu,n}$. To prove the second part of this lemma, it is sufficient to show that the \mathcal{N}_{k_2} -seminorms of $\tilde{p}_{\mathbf{n}\nu,n}$ and $\tilde{r}_{\mathbf{n}\nu,n}$ are bounded by a term that does not depend on ν . Indeed, by the formula of $\tilde{P}_{\nu,n}$ in (4.11), the \hbar -scaled symbol calculus and Egorov's theorem, we find that the \mathcal{N}_{k_2} -seminorm of $\tilde{p}_{\mathbf{n}\nu,n}$ depends on \mathcal{N}_{k_2} -seminorm of λ_s (which is independent of ν already) and \mathcal{N}_{k_2+2} -seminorm of $S_{\mathbf{n}\nu,n}$; and the \mathcal{N}_{k_2} -seminorm of $\tilde{r}_{\mathbf{n}\nu,n}$ depends on \mathcal{N}_{k_2+2} -seminorm of λ_s and \mathcal{N}_{k_2+4} -seminorm of $S_{\mathbf{n}\nu,n}$. It all boils down to showing that the \mathcal{N}_{k_2+2} - and \mathcal{N}_{k_2+4} -seminorms of $S_{\mathbf{n}\nu,n}$ are bounded by a term irrelevant to ν , which easily follows from the formula of $S_{\mathbf{n}\nu,n}$ given in (4.9) and a similar argument as in (4.17). \square

5. ESTIMATES OF $\tilde{p}_{\mathbf{n}\nu,n}$: PROOF OF LEMMA 5.1

5.1. Lemma 5.1 and Its Proof. The estimates given in Lemma 5.1 in this section is the most important step to prove Proposition 3.2.

Lemma 5.1. *If $\tau(x)$ is not an essential coboundary over T , then there exists $n_0 \in \mathbb{N}$ such that*

$$(5.1) \quad \tilde{p}_0 := \sup_{\nu \in \mathbb{Z}^\ell \setminus \{\mathbf{0}\}} \sup_{(x, \xi) \in T^* \mathbb{T}^d} \tilde{p}_{\mathbf{n}\nu, n_0}(x, \xi) < 1.$$

Proof. Given $\mathbf{n} \in \mathbb{S}^{\ell-1}$ and $x \in \mathbb{T}^d$, we consider the affine map $\mathcal{F}_{\mathbf{n},x} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ given by

$$(5.2) \quad \mathcal{F}_{\mathbf{n},x}(\xi) = (D_x T)^t \xi + (D_x \tau)^t \mathbf{n} \quad \text{for any } \xi \in \mathbb{R}^d,$$

and the n -th iterates

$$(5.3) \quad \mathcal{F}_{\mathbf{n},x}^n(\xi) = \prod_{k=0}^{n-1} \mathcal{F}_{\mathbf{n},T^k x}(\xi) = (D_x T^n)^t \xi + W_{\mathbf{n},n}(x) \quad \text{for any } n \in \mathbb{N},$$

where $W_{\mathbf{n},n}(x) = W_n(x)\mathbf{n}$, and $W_n(x)$ is given by (4.10). Conventionally, we set $W_{\mathbf{n},0}(x) = \mathbf{0}$ and $\mathcal{F}_{\mathbf{n},x}^0 = id$. We also define

$$(5.4) \quad \tilde{p}_{\mathbf{n},n}(x, \xi) = \sum_{x=T^n y} \mathcal{A}_n(y) \left[\frac{g(\mathcal{F}_{\mathbf{n},y}^n(\xi))}{g(\xi)} \right]^{2s},$$

which extends the formula of $\tilde{p}_{\mathbf{n},n}(x, \xi)$ given by (4.18) to all $\mathbf{n} \in \mathbb{S}^{\ell-1}$. For any fixed $n \in \mathbb{N}$, it is easy to check that the function $(\mathbf{n}, x, \xi) \mapsto \tilde{p}_{\mathbf{n},n}(x, \xi)$ is of class C^∞ . Further, we set

$$(5.5) \quad \tilde{p}_{\mathbf{n},n} := \sup_{(x, \xi) \in T^* \mathbb{T}^d} \tilde{p}_{\mathbf{n},n}(x, \xi).$$

The properties of $\tilde{p}_{\mathbf{n},n}$ will be given by Sublemma 5.3 below.

Recall that R is given by (4.1). By Sublemma 5.4 below, for any $\mathbf{n} \in \mathbb{S}^{\ell-1}$, there exists $n_0(\mathbf{n}) \in \mathbb{N}$ such that for any $(x, \xi) \in T^* \mathbb{T}^d$, we have $|\mathcal{F}_{\mathbf{n},y}^{n_0(\mathbf{n})}(\xi)| > 2R$ for some $y \in T^{-n_0(\mathbf{n})}(x)$. Then by (5.3) and (4.10), we have for any $\mathbf{n}' \in \mathbb{S}^{\ell-1}$,

$$\begin{aligned} |\mathcal{F}_{\mathbf{n}',y}^{n_0(\mathbf{n})}(\xi)| &\geq |\mathcal{F}_{\mathbf{n},y}^{n_0(\mathbf{n})}(\xi)| - |\mathcal{F}_{\mathbf{n}',y}^{n_0(\mathbf{n})}(\xi) - \mathcal{F}_{\mathbf{n},y}^{n_0(\mathbf{n})}(\xi)| \\ &\geq 2R - |W_{n_0(\mathbf{n})}(y)| |\mathbf{n}' - \mathbf{n}| \\ &\geq 2R - \|DT\| \|D\tau\| n_0(\mathbf{n}) |\mathbf{n}' - \mathbf{n}|. \end{aligned}$$

Hence there is $\varepsilon(\mathbf{n}) > 0$ such that $|\mathcal{F}_{\mathbf{n}',y}^{n_0(\mathbf{n})}(\xi)| > R$ whenever $|\mathbf{n}' - \mathbf{n}| < \varepsilon(\mathbf{n})$. By Sublemma 5.2 (2), we get $\tilde{p}_{\mathbf{n}',n_0(\mathbf{n})}(x, \xi) < 1$. Together with Sublemma 5.2 (3),

$$\begin{aligned} \tilde{p}_{\mathbf{n}',n_0(\mathbf{n})} &= \max \left\{ \sup_{(x, \xi) \in \mathcal{U}_R} \tilde{p}_{\mathbf{n}',n_0(\mathbf{n})}(x, \xi), \sup_{(x, \xi) \in T^* \mathbb{T}^d \setminus \mathcal{U}_R} \tilde{p}_{\mathbf{n}',n_0(\mathbf{n})}(x, \xi) \right\} \\ &\leq \max \left\{ \max_{(x, \xi) \in \mathcal{U}_R} \tilde{p}_{\mathbf{n}',n_0(\mathbf{n})}(x, \xi), \left(\frac{\gamma + 1}{2} \right)^{2s} \right\} < 1, \end{aligned}$$

where $\mathcal{U}_R := \{(x, \xi) : |\xi| \leq R\}$ is a compact subdomain in $T^* \mathbb{T}^d$, and γ is given by (4.1). Moreover, by Sublemma 5.3 (2), $\tilde{p}_{\mathbf{n}',n} \leq \tilde{p}_{\mathbf{n}',n_0(\mathbf{n})} < 1$ for any $n \geq n_0(\mathbf{n})$.

To sum up, for any $\mathbf{n} \in \mathbb{S}^{\ell-1}$, there are $n_0(\mathbf{n}) \in \mathbb{N}$ and $\varepsilon(\mathbf{n}) > 0$ such that $\tilde{p}_{\mathbf{n}',n} < 1$ for all $\mathbf{n}' \in B(\mathbf{n}, \varepsilon(\mathbf{n}))$ and $n \geq n_0(\mathbf{n})$, where $B(\mathbf{n}, \varepsilon(\mathbf{n}))$ denotes the open ball in $\mathbb{S}^{\ell-1}$ with center at \mathbf{n} and of radius $\varepsilon(\mathbf{n})$. Since $\mathbb{S}^{\ell-1}$ is compact, there are $\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_k \in \mathbb{S}^{\ell-1}$ such that the finite collection of open balls $\{B(\mathbf{n}_j, \varepsilon(\mathbf{n}_j))\}_{1 \leq j \leq k}$ covers $\mathbb{S}^{\ell-1}$. Therefore, if we set

$$n_0 = \max\{n_0(\mathbf{n}_1), \dots, n_0(\mathbf{n}_k)\},$$

then $\tilde{p}_{\mathbf{n},n_0} < 1$ for all $\mathbf{n} \in \mathbb{S}^{\ell-1}$. By Sublemma 5.3 (3), we know that the function $\mathbf{n} \mapsto \tilde{p}_{\mathbf{n},n_0}$ is continuous, then $\sup_{\mathbf{n} \in \mathbb{S}^{\ell-1}} \tilde{p}_{\mathbf{n},n_0} = \max_{\mathbf{n} \in \mathbb{S}^{\ell-1}} \tilde{p}_{\mathbf{n},n_0} < 1$, from which (5.1) follows. \square

5.2. Sublemmas and Proofs.

Sublemma 5.2. *Recall that R and γ are given by (4.1) and (1.1) respectively. Let $\mathbf{n} \in \mathbb{S}^{\ell-1}$. We have the following:*

- (1) $\tilde{p}_{\mathbf{n},n}(x, \xi) \leq 1$ for all $n \in \mathbb{N}$ and $(x, \xi) \in T^*\mathbb{T}^d$;
- (2) $\tilde{p}_{\mathbf{n},n}(x, \xi) < 1$ if and only if there is $y \in T^{-n}x$ such that $|\mathcal{F}_{\mathbf{n},y}^n(\xi)| > R$;
- (3) $\tilde{p}_{\mathbf{n},n}(x, \xi) \leq \left(\frac{\gamma+1}{2}\right)^{2s} < 1$ for all $n \in \mathbb{N}$ and $(x, \xi) \in T^*\mathbb{T}^d$ with $|\xi| > R$.

Proof. The key observation is the following: for any $n \in \mathbb{N}$, $y \in \mathbb{T}^d$ and $\xi \in \mathbb{R}^d$,

$$(5.6) \quad |\mathcal{F}_{\mathbf{n},y}^n(\xi)| > \frac{\gamma+1}{2}|\xi|, \quad \text{if } |\xi| > R.$$

Indeed, by the choice of R in (4.1), we have $|\xi| > R > \frac{2\|D\tau\|}{\gamma-1}$. So by (5.2),

$$|\mathcal{F}_{\mathbf{n},y}^n(\xi)| \geq |(D_y T)^t \xi| - |(D_y \tau)^t \mathbf{n}| \geq \gamma|\xi| - \|D\tau\| \geq \gamma|\xi| - \frac{\gamma-1}{2}|\xi| = \frac{\gamma+1}{2}|\xi|.$$

Hence by induction, we have for all $n \geq 1$,

$$|\mathcal{F}_{\mathbf{n},y}^n(\xi)| \geq \left(\frac{\gamma+1}{2}\right)^n |\xi| \geq \frac{\gamma+1}{2}|\xi|.$$

Consequently, for any $n \in \mathbb{N}$, $y \in \mathbb{T}^d$ and $\xi \in \mathbb{R}^d$,

- if $|\mathcal{F}_{\mathbf{n},y}^n(\xi)| \leq R$, then $|\mathcal{F}_{\mathbf{n},y}^k(\xi)| \leq R$ for all $0 \leq k \leq n$, and in particular, we must have $|\xi| \leq R$;
- if $|\mathcal{F}_{\mathbf{n},y}^n(\xi)| > R$, then $|\mathcal{F}_{\mathbf{n},y}^n(\xi)| > |\xi|$ no matter whether $|\xi| > R$ or not.

By the definition of $g(\xi)$ given by (4.2), the quotient

$$\left[\frac{g(\mathcal{F}_{\mathbf{n},y}^n(\xi))}{g(\xi)} \right]^{2s} \begin{cases} = 1 & \text{if } |\mathcal{F}_{\mathbf{n},y}^n(\xi)| \leq R; \\ < 1 & \text{otherwise.} \end{cases}$$

In either case, we always get

$$(5.7) \quad 0 < \left[\frac{g(\mathcal{F}_{\mathbf{n},y}^n(\xi))}{g(\xi)} \right]^{2s} \leq 1.$$

Recall that $\sum_{x=T^n y} \mathcal{A}_n(y) = 1$, where $\mathcal{A}_n(y)$ is positive and defined by (3.5). Therefore, for any $n \in \mathbb{N}$ and $(x, \xi) \in T^*\mathbb{T}^d$,

$$\tilde{p}_{\mathbf{n},n}(x, \xi) = \sum_{x=T^n y} \mathcal{A}_n(y) \left[\frac{g(\mathcal{F}_{\mathbf{n},y}^n(\xi))}{g(\xi)} \right]^{2s} \leq \sum_{x=T^n y} \mathcal{A}_n(y) = 1.$$

Clearly, we have that $\tilde{p}_{\mathbf{n},n}(x, \xi) < 1$ if and only if $|\mathcal{F}_{\mathbf{n},y}^n(\xi)| > R$ for some $y \in T^{-n}x$. Moreover, if $|\xi| > R$, then by (5.6), we have that

$$\left[\frac{g(\mathcal{F}_{\mathbf{n},y}^n(\xi))}{g(\xi)} \right]^{2s} \leq \left(\frac{\gamma+1}{2} \right)^{2s} \quad \text{for all } y \in T^{-n}x,$$

and hence $\tilde{p}_{\mathbf{n},n}(x, \xi) \leq \left(\frac{\gamma+1}{2}\right)^{2s}$. □

Sublemma 5.3. *Let $\tilde{p}_{\mathbf{n},n}$ be defined as in (5.5).*

- (1) *For any $\mathbf{n} \in \mathbb{S}^{\ell-1}$ and $n \in \mathbb{N}$, we have that $0 < \tilde{p}_{\mathbf{n},n} \leq 1$;*

- (2) For any $\mathbf{n} \in \mathbb{S}^{\ell-1}$, the sequence $\{\tilde{p}_{\mathbf{n},n}\}_{n \in \mathbb{N}}$ is non-increasing;
(3) For any $n \in \mathbb{N}$, the function $\mathbf{n} \mapsto \tilde{p}_{\mathbf{n},n}$ is continuous.

Proof. As shown by Sublemma 5.2 (1), $0 < \tilde{p}_{\mathbf{n},n}(x, \xi) \leq 1$ for any $\mathbf{n} \in \mathbb{S}^{\ell-1}$, $n \in \mathbb{N}$ and $(x, \xi) \in T^* \mathbb{T}^d$, and thus $0 < \tilde{p}_{\mathbf{n},n} \leq 1$.

Let $\mathbf{n} \in \mathbb{S}^{\ell-1}$ be fixed. For any $(x, \xi) \in T^* \mathbb{T}^d$ and $n, m \in \mathbb{N}$, by (5.7) we have

$$\begin{aligned}
\tilde{p}_{\mathbf{n},n+m}(x, \xi) &= \sum_{x=T^{n+m}y} \mathcal{A}_{n+m}(y) \left[\frac{g(\mathcal{F}_{\mathbf{n},y}^{n+m}(\xi))}{g(\xi)} \right]^{2s} \\
&= \sum_{x=T^n z} \sum_{z=T^m y} \mathcal{A}_n(z) \mathcal{A}_m(y) \left[\frac{g(\mathcal{F}_{\mathbf{n},y}^m(\xi))}{g(\xi)} \right]^{2s} \left[\frac{g(\mathcal{F}_{\mathbf{n},z}^n(\mathcal{F}_{\mathbf{n},y}^m(\xi)))}{g(\mathcal{F}_{\mathbf{n},y}^m(\xi))} \right]^{2s} \\
&\leq \sum_{x=T^n z} \mathcal{A}_n(z) \sum_{z=T^m y} \mathcal{A}_m(y) \left[\frac{g(\mathcal{F}_{\mathbf{n},y}^m(\xi))}{g(\xi)} \right]^{2s} \\
&= \sum_{x=T^n z} \mathcal{A}_n(z) \tilde{p}_{\mathbf{n},m}(z, \xi) \leq \tilde{p}_{\mathbf{n},m} \sum_{x=T^n z} \mathcal{A}_n(z) = \tilde{p}_{\mathbf{n},m}.
\end{aligned}$$

Hence, $\tilde{p}_{\mathbf{n},n+m} = \sup_{(x, \xi) \in T^* \mathbb{T}^d} \tilde{p}_{\mathbf{n},n+m}(x, \xi) \leq \tilde{p}_{\mathbf{n},m}$. This proves that the sequence $\{\tilde{p}_{\mathbf{n},n}\}_{n \in \mathbb{N}}$ is non-increasing.

Let $n \in \mathbb{N}$ be fixed. To show that the function $\mathbf{n} \mapsto \tilde{p}_{\mathbf{n},n}$ is continuous, it suffices to show that the family

$$\{\mathbf{n} \mapsto \tilde{p}_{\mathbf{n},n}(x, \xi) : (x, \xi) \in T^* \mathbb{T}^d\} \subset C^0(\mathbb{S}^{\ell-1})$$

is uniformly bounded and equicontinuous. The uniform boundedness is already given by Sublemma 5.2 (1). The equicontinuity follows from that

$$\begin{aligned}
&\sup_{(x, \xi) \in T^* \mathbb{T}^d} \left| \frac{\partial}{\partial \mathbf{n}} \tilde{p}_{\mathbf{n},n}(x, \xi) \right| \\
&= \sup_{(x, \xi) \in T^* \mathbb{T}^d} \left| \sum_{x=T^n y} \mathcal{A}_n(y) \frac{2s [g(\mathcal{F}_{\mathbf{n},y}^n(\xi))]^{2s-1}}{[g(\xi)]^{2s}} [Dg(\mathcal{F}_{\mathbf{n},y}^n(\xi))]^t W_n(y) \right| \\
&\leq 2|s| \sup_{(x, \xi) \in T^* \mathbb{T}^d} \sum_{x=T^n y} \mathcal{A}_n(y) \left[\frac{g(\mathcal{F}_{\mathbf{n},y}^n(\xi))}{g(\xi)} \right]^{2s} \frac{\|Dg\| \cdot n \|DT\| \|D\tau\|}{g(\mathcal{F}_{\mathbf{n},y}^n(\xi))} \\
&\leq 2n|s| \|Dg\| \|DT\| \|D\tau\| \tilde{p}_{\mathbf{n},n} \leq 2n|s| \|Dg\| \|DT\| \|D\tau\| < \infty.
\end{aligned}$$

Here we have used (5.3), (5.4), (4.10) and the following properties of $g(\xi)$ (see (4.2)): $g(\xi) \geq 1$ for any $\xi \in \mathbb{R}^d$ and $\|Dg\| = \sup_{\xi \in \mathbb{R}^d} |Dg(\xi)| < \infty$. \square

Sublemma 5.4. Suppose $\tau(x)$ is not an essential coboundary over T . For any $\mathbf{n} \in \mathbb{S}^{\ell-1}$, there is $n_0(\mathbf{n}) \in \mathbb{N}$ such that for any $(x, \xi) \in T^* \mathbb{T}^d$, $|\mathcal{F}_{\mathbf{n},y}^{n_0(\mathbf{n})}(\xi)| > 2R$ for some $y \in T^{-n_0(\mathbf{n})}(x)$.

Proof. Let us argue by contradiction. If this sublemma does not hold for some $\mathbf{n}^* \in \mathbb{S}^{\ell-1}$, then for any $n \in \mathbb{N}$, there is $(x_n, \xi_n) \in T^* \mathbb{T}^d$ such that $|\mathcal{F}_{\mathbf{n}^*,y}^n(\xi_n)| \leq 2R$ for any $y \in T^{-n}(x_n)$. In fact, we further have

$$(5.8) \quad |\mathcal{F}_{\mathbf{n}^*,y}^k(\xi_n)| \leq 2R, \quad \text{for any } y \in T^{-n}(x_n) \text{ and } 0 \leq k \leq n,$$

since otherwise if $|\mathcal{F}_{\mathbf{n}^*,y}^k(\xi_n)| > 2R$ for some $0 \leq k < n$, then by (5.6), $|\mathcal{F}_{\mathbf{n}^*,y}^n(\xi_n)| = |\mathcal{F}_{\mathbf{n}^*,T^ky}^{n-k}(\mathcal{F}_{\mathbf{n}^*,y}^k(\xi_n))| \geq \frac{\gamma+1}{2}|\mathcal{F}_{\mathbf{n}^*,y}^k(\xi_n)| > 2R$. Note that in particular, $|\xi_n| \leq 2R$.

Using (5.3), we rewrite (5.8) as

$$|\mathcal{F}_{\mathbf{n}^*,y}^k(\xi_n)| = |(D_y T^k)^t \xi_n + W_{\mathbf{n}^*,k}(y)| = \left| (D_y T^k)^t \left(\xi_n + \widetilde{W}_{\mathbf{n}^*,k}(y) \right) \right| \leq 2R,$$

for any $y \in T^{-n}(x_n)$ and $0 \leq k \leq n$, where

$$(5.9) \quad \widetilde{W}_{\mathbf{n}^*,k}(y) = [(D_y T^k)^t]^{-1} W_{\mathbf{n}^*,k}(y) = \sum_{j=0}^{k-1} [(D_{T^j y} T^{k-j})^t]^{-1} [D_{T^j y} \tau]^t \mathbf{n}^*.$$

By (1.1), we have

$$(5.10) \quad \left| \xi_n + \widetilde{W}_{\mathbf{n}^*,k}(y) \right| \leq \frac{2R}{\gamma^k}, \quad \text{for any } y \in T^{-n}(x_n), \quad 0 \leq k \leq n.$$

We would like to rewrite $\widetilde{W}_{\mathbf{n}^*,k}(y)$ in terms of x_n as follows. Suppose the degree of the expanding endomorphism $T : \mathbb{T}^d \rightarrow \mathbb{T}^d$ is N . We denote

$$\Sigma_N^n = \{\mathbf{i} = (i_1, i_2, \dots, i_n) : i_j = 0, 1, \dots, N-1, \quad 1 \leq n \leq \infty\}.$$

Let $T_0^{-1}, T_1^{-1}, \dots, T_{N-1}^{-1}$ be the inverse branches of T . Given $x \in \mathbb{T}^d$ and $\mathbf{i} \in \Sigma_N^n$, we denote $T_{\mathbf{i}}^{-j} x = T_{i_j}^{-1} \dots T_{i_1}^{-1} x$, which is well-defined whenever $0 \leq j \leq n \leq \infty$ and j is finite. We then define

$$(5.11) \quad V_{\mathbf{n}^*,k}(\mathbf{i}, x) := \sum_{j=1}^k D_x \left[\tau(T_{\mathbf{i}}^{-j}(x)) \cdot \mathbf{n}^* \right] = \sum_{j=1}^k [(D_{T_{\mathbf{i}}^{-j} x} T^j)^t]^{-1} (D_{T_{\mathbf{i}}^{-j} x} \tau)^t \mathbf{n}^*.$$

for any $1 \leq k \leq n \leq \infty$ and k is finite. Note that

$$(5.12) \quad \sum_{j=m}^{\infty} \left| [(D_{T_{\mathbf{i}}^{-j} x} T^j)^t]^{-1} [D_{T_{\mathbf{i}}^{-j} x} \tau]^t \mathbf{n}^* \right| \leq \|D\tau\| \sum_{j=m}^{\infty} \gamma^{-j} \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

and the convergence is uniform. That is, the sequence $\{V_{\mathbf{n}^*,k}(\mathbf{i}, x)\}_1^{\infty}$ is uniformly Cauchy. Hence $V_{\mathbf{n}^*,\infty}(\mathbf{i}, x)$ is well-defined as in (5.11) for all $x \in \mathbb{T}^d$ and $\mathbf{i} \in \Sigma_N^{\infty}$. Denote $V_{\mathbf{n}^*}(\mathbf{i}, x) = V_{\mathbf{n}^*,\infty}(\mathbf{i}, x)$. We have

$$(5.13) \quad \lim_{k \rightarrow \infty} V_{\mathbf{n}^*,k}(\mathbf{i}, x) = V_{\mathbf{n}^*}(\mathbf{i}, x) \quad \text{for any } x \in \mathbb{T}^d, \quad \mathbf{i} \in \Sigma_N^{\infty},$$

and the convergence is uniform.

Comparing (5.9) and (5.11), we see $\widetilde{W}_{\mathbf{n}^*,k}(y) = V_{\mathbf{n}^*,k}(\mathbf{i}, x)$ whenever $y = T_{\mathbf{i}}^{-n}(x)$. Therefore, we can rewrite (5.10) as

$$|\xi_n + V_{\mathbf{n}^*,k}(\mathbf{i}, x_n)| \leq \frac{2R}{\gamma^k}$$

for any $\mathbf{i} \in \Sigma_N^n$ and $1 \leq k \leq n$. Since the sequence $\{(x_n, \xi_n)\}$ lies in the compact subdomain $\mathcal{U}_{2R} := \{(x, \xi) : |\xi| \leq 2R\}$ of $T^* \mathbb{T}^d$, there is an accumulation point (x^*, ξ^*) . Choosing subsequences if necessary, we take $n \rightarrow \infty$ firstly and $k \rightarrow \infty$ secondly in the above inequality, then we obtain $V_{\mathbf{n}^*}(\mathbf{i}, x^*) = -\xi^*$, regardless of the choice for $\mathbf{i} \in \Sigma_N^{\infty}$.

For any $x \in \mathbb{T}^d$, take $w \in \{0, 1, \dots, N-1\}$ such that $x = T_w^{-1}(Tx)$. For any $\mathbf{i} \in \Sigma_N^{\infty}$, we can directly check (5.11) (when $k = \infty$) to get

$$(5.14) \quad (D_x T)^t V_{\mathbf{n}^*}(w\mathbf{i}, Tx) = V_{\mathbf{n}^*}(\mathbf{i}, x) + (D_x \tau)^t \mathbf{n}^*.$$

By Claim 1 below, we know that $V_{\mathbf{n}^*}(\mathbf{i}, x)$ is independent of \mathbf{i} for any $x \in \mathbb{T}^d$. Hence, we can define a function $V_{\mathbf{n}^*} : \mathbb{T}^d \rightarrow \mathbb{R}^d$ by

$$V_{\mathbf{n}^*}(x) = V_{\mathbf{n}^*}(\mathbf{i}, x), \quad \text{for any } \mathbf{i} \in \Sigma_N^\infty,$$

and thus (5.14) is rewritten as

$$(5.15) \quad (D_x T)^t V_{\mathbf{n}^*}(Tx) = V_{\mathbf{n}^*}(x) + (D_x \tau)^t \mathbf{n}^*.$$

By Claim 2 below, which asserts that the 1-form on \mathbb{T}^d given by $V_{\mathbf{n}^*}(x) \cdot dx$ is exact, there is a potential function u such that $\nabla_x u = V_{\mathbf{n}^*}(x)$. Alternatively, we can define the function $u : \mathbb{T}^d \rightarrow \mathbb{R}$ by

$$u(x) = \int_{\Gamma_{\mathbf{0}, x}} V_{\mathbf{n}^*}(z) \cdot dz, \quad x \in \mathbb{T}^d,$$

where $\Gamma_{\mathbf{0}, x}$ is any smooth path in \mathbb{T}^d from $\mathbf{0} = (0, 0, \dots, 0)$ to x .

On both sides of (5.15), we replace x by tx , take the dot product with x and integrate with respect to t from 0 to 1, then we get

$$\int_{\Gamma_{T\mathbf{0}, Tx}^1} V_{\mathbf{n}^*}(z) \cdot dz = \int_{\Gamma_{\mathbf{0}, x}^0} V_{\mathbf{n}^*}(z) \cdot dz + \int_{\Gamma_{\mathbf{0}, x}^0} (D_z \tau)^t \mathbf{n}^* \cdot dz,$$

where $\Gamma_{\mathbf{0}, x}^0 := \{tx : 0 \leq t \leq 1\}$, and $\Gamma_{T\mathbf{0}, Tx}^1 := \{T(tx) : 0 \leq t \leq 1\}$. In other words, we have

$$u(Tx) - u(T\mathbf{0}) = u(x) - u(\mathbf{0}) + \mathbf{n}^* \cdot \tau(x) - \mathbf{n}^* \cdot \tau(\mathbf{0}).$$

Note that $u(\mathbf{0}) = 0$. Let $c = \mathbf{n}^* \cdot \tau(\mathbf{0}) - u(T\mathbf{0})$, then we get

$$\mathbf{n}^* \cdot \tau(x) = c - u(x) + u(Tx),$$

which contradicts to the fact that $\tau(x)$ is not an essential coboundary over T . \square

Claim 1. For any $x \in \mathbb{T}^d$, $V_{\mathbf{n}^*}(\mathbf{i}, x)$ is independent of \mathbf{i} , that is, $V_{\mathbf{n}^*}(\mathbf{i}, x) = V_{\mathbf{n}^*}(\mathbf{i}', x)$ for all $\mathbf{i}, \mathbf{i}' \in \Sigma_N^\infty$.

Proof. Recall that $V_{\mathbf{n}^*}(\mathbf{i}, x^*) = -\xi^*$ for any $\mathbf{i} \in \Sigma_N^\infty$, where (x^*, ξ^*) is an accumulation point of the sequence (x_n, ξ_n) .

Taking $x = T_w^{-1}x^*$ in (5.14) for some $w \in \{0, 1, \dots, N-1\}$, we get

$$V_{\mathbf{n}^*}(\mathbf{i}, T_w^{-1}x^*) = -(D_{T_w^{-1}x^*} T)^t \xi^* - [D_{T_w^{-1}x^*} \tau]^t \mathbf{n}^*.$$

The right hand side is independent of \mathbf{i} , and hence $V_{\mathbf{n}^*}(\mathbf{i}, T_w^{-1}x^*) = V_{\mathbf{n}^*}(\mathbf{0}, T_w^{-1}x^*)$, where $\mathbf{0} = (0, 0, \dots) \in \Sigma_N^\infty$.

Inductively, one can show that $V_{\mathbf{n}^*}(\mathbf{i}, x) = V_{\mathbf{n}^*}(\mathbf{0}, x)$ for all $x \in \bigcup_{n=1}^\infty T^{-n}(x^*)$ and thus for all $x \in \mathbb{T}^d$, since the set $\bigcup_{n=1}^\infty T^{-n}x^*$ is dense in \mathbb{T}^d . \square

Claim 2. The 1-form on \mathbb{T}^d given by

$$V_{\mathbf{n}^*}(x) \cdot dx = V_{\mathbf{n}^*}^1(x) dx_1 + \dots + V_{\mathbf{n}^*}^d(x) dx_d$$

is exact.

Proof. We first show that $V_{\mathbf{n}^*}(x) \cdot dx$ is a closed 1-form, which is equivalent to showing that for any $x \in \mathbb{T}^d$,

$$(5.16) \quad \frac{\partial}{\partial x_i} V_{\mathbf{n}^*}^j(x) = \frac{\partial}{\partial x_j} V_{\mathbf{n}^*}^i(x), \quad 1 \leq i \leq j \leq d.$$

Indeed, by (5.13) and Claim 1, $V_{\mathbf{n}^*,k}^j(\mathbf{i}, x)$ converges uniformly to $V_{\mathbf{n}^*}^j(\mathbf{i}, x) = V_{\mathbf{n}^*}^j(x)$ as $k \rightarrow \infty$. By similar calculation as in (5.12), we have that $\frac{\partial}{\partial x_i} V_{\mathbf{n}^*,k}^j(\mathbf{i}, x)$ converges uniformly as $k \rightarrow \infty$, and hence $\frac{\partial}{\partial x_i} V_{\mathbf{n}^*}^j(x) = \lim_{k \rightarrow \infty} \frac{\partial}{\partial x_i} V_{\mathbf{n}^*,k}^j(\mathbf{i}, x)$. We see from (5.11) that for each $k \in \mathbb{N}$ and any $\mathbf{i} \in \Sigma_N^\infty$, the 1-form $V_{\mathbf{n}^*,k}(\mathbf{i}, x) \cdot dx = d\left(\sum_{j=1}^k \tau(T_{\mathbf{i}}^{-j}(x)) \cdot \mathbf{n}^*\right)$ is exact and hence closed. Thus,

$$\frac{\partial}{\partial x_i} V_{\mathbf{n}^*,k}^j(\mathbf{i}, x) = \frac{\partial}{\partial x_j} V_{\mathbf{n}^*,k}^i(\mathbf{i}, x), \quad 1 \leq i \leq j \leq d,$$

from which (5.16) follows by taking $k \rightarrow \infty$.

Now to show that $V_{\mathbf{n}^*}(x) \cdot dx$ is exact. Since $V_{\mathbf{n}^*}(x) \cdot dx$ is closed, it is sufficient to prove that for any $x = (x_1, x_2, \dots, x_d) \in \mathbb{T}^d$ and $1 \leq k \leq d$,

$$(5.17) \quad \int_0^1 V_{\mathbf{n}^*}^k(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_d) dt = 0.$$

To see this, by (5.11) and Claim 1, we rewrite for arbitrary $M \in \mathbb{N}$,

$$\begin{aligned} V_{\mathbf{n}^*}(x) &= \frac{1}{N^M} \sum_{\mathbf{i} \in \Sigma_N^M} V_{\mathbf{n}^*}(\mathbf{i}\mathbf{0}, x) \\ &= \frac{1}{N^M} \sum_{\mathbf{i} \in \Sigma_N^M} V_{\mathbf{n}^*,M}(\mathbf{i}\mathbf{0}, x) + \frac{1}{N^M} \sum_{\mathbf{i} \in \Sigma_N^M} [V_{\mathbf{n}^*}(\mathbf{i}\mathbf{0}, x) - V_{\mathbf{n}^*,M}(\mathbf{i}\mathbf{0}, x)] \\ &= \frac{1}{N^M} \sum_{j=1}^M \sum_{\mathbf{i} \in \Sigma_N^j} D_x \left[\tau(T_{i_j}^{-1} \dots T_{i_1}^{-1}(x)) \cdot \mathbf{n}^* \right] + \frac{1}{N^M} \sum_{\mathbf{i} \in \Sigma_N^M} [V_{\mathbf{n}^*}(\mathbf{i}\mathbf{0}, x) - V_{\mathbf{n}^*,M}(\mathbf{i}\mathbf{0}, x)] \\ &=: I_{\mathbf{n}^*}(x) + J_{\mathbf{n}^*}(x). \end{aligned}$$

On one hand, let $I_{\mathbf{n}^*}^k$ be the k -th component of $I_{\mathbf{n}^*}$, then

$$\begin{aligned} &\int_0^1 I_{\mathbf{n}^*}^k(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_d) dt \\ &= \frac{1}{N^M} \sum_{j=1}^M \sum_{\mathbf{i} \in \Sigma_N^j} \int_0^1 \frac{\partial}{\partial x_k} \left[\tau(T_{i_j}^{-1} \dots T_{i_1}^{-1}(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_d)) \cdot \mathbf{n}^* \right] dt \\ &= \frac{1}{N^M} \sum_{j=1}^M \mathbf{n}^* \cdot \sum_{\mathbf{i} \in \Sigma_N^j} \left[\tau(T_{i_j}^{-1} \dots T_{i_1}^{-1}(x_1, \dots, x_{k-1}, 1, x_{k+1}, \dots, x_d)) \right. \\ &\quad \left. - \tau(T_{i_j}^{-1} \dots T_{i_1}^{-1}(x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_d)) \right] \\ &= \frac{1}{N^M} \sum_{j=1}^M \mathbf{n}^* \cdot \left[\sum_{\mathbf{i} \in \Sigma_N^j} \tau(T_{i_j}^{-1} \dots T_{i_1}^{-1}(x_1, \dots, x_{k-1}, 1, x_{k+1}, \dots, x_d)) \right. \\ &\quad \left. - \sum_{\mathbf{i} \in \Sigma_N^j} \tau(T_{i_j}^{-1} \dots T_{i_1}^{-1}(x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_d)) \right] = 0. \end{aligned}$$

The last term must vanish since $\{T_{i_j}^{-1} \dots T_{i_1}^{-1}(x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_d) : \mathbf{i} \in \Sigma_N^j\}$ and $\{T_{i_j}^{-1} \dots T_{i_1}^{-1}(x_1, \dots, x_{k-1}, 1, x_{k+1}, \dots, x_d) : \mathbf{i} \in \Sigma_N^j\}$ are just two representations for the set of all j -th pre-images of the point $(x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_d) = (x_1, \dots, x_{k-1}, 1, x_{k+1}, \dots, x_d)$ in \mathbb{T}^d .

On the other hand, by (5.12) and (5.13), the convergence $V_{\mathbf{n}*,M}(\mathbf{i}\mathbf{0}, x) \rightarrow V_{\mathbf{n}*}(\mathbf{i}\mathbf{0}, x)$ is uniform in \mathbf{i} and x as $M \rightarrow \infty$. By choosing M large enough, the integral of the k -th component of $J_{\mathbf{n}*}(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_d)$ with respect to t from 0 to 1 is arbitrary small and hence 0. It follows that (5.17) holds. \square

REFERENCES

- [1] Jean-François Arnoldi. Fractal Weyl law for skew extensions of expanding maps. *Nonlinearity*, 25(6):1671–1693, 2012.
- [2] Jean-François Arnoldi, Frédéric Faure, and Tobias Weich. Asymptotic spectral gap and Weyl law for ruelle resonances of open partially expanding maps. *Preprint*, 2013.
- [3] Artur Avila, Sébastien Gouëzel, and Jean-Christophe Yoccoz. Exponential mixing for the Teichmüller flow. *Publ. Math. Inst. Hautes Études Sci.*, (104):143–211, 2006.
- [4] Viviane Baladi and Carlangelo Liverani. Exponential decay of correlations for piecewise cone hyperbolic contact flows. *Comm. Math. Phys.*, 314(3):689–773, 2012.
- [5] Viviane Baladi and Brigitte Vallée. Exponential decay of correlations for surface semi-flows without finite Markov partitions. *Proc. Amer. Math. Soc.*, 133(3):865–874 (electronic), 2005.
- [6] Keith Burns and Amie Wilkinson. Stable ergodicity of skew products. *Ann. Sci. École Norm. Sup. (4)*, 32(6):859–889, 1999.
- [7] Oliver Butterley and Peyman Eslami. Exponential mixing for skew products with discontinuities. *Preprint*, 2014.
- [8] Dmitry Dolgopyat. On decay of correlations in anosov flows. *Ann. of Math. (2)*, 147:357–390, 1998.
- [9] Dmitry Dolgopyat. On mixing properties of compact group extensions of hyperbolic systems. *Israel J. Math.*, 130:157–205, 2002.
- [10] Yu. V. Egorov. *Linear differential equations of principal type*. Contemporary Soviet Mathematics. Consultants Bureau, New York, 1986. Translated from the Russian by Dang Prem Kumar.
- [11] Frédéric Faure. Prequantum chaos: resonances of the prequantum cat map. *J. Mod. Dyn.*, 1(2):255–285, 2007.
- [12] Frédéric Faure. Semiclassical origin of the spectral gap for transfer operators of a partially expanding map. *Nonlinearity*, 24(5):1473–1498, 2011.
- [13] Frédéric Faure and Nicolas Roy. Ruelle-Pollicott resonances for real analytic hyperbolic maps. *Nonlinearity*, 19(6):1233–1252, 2006.
- [14] Frédéric Faure, Nicolas Roy, and Johannes Sjöstrand. Semi-classical approach for Anosov diffeomorphisms and Ruelle resonances. *Open Math. J.*, 1:35–81, 2008.
- [15] Michael Field, Ian Melbourne, and Andrei Török. Stable ergodicity for smooth compact Lie group extensions of hyperbolic basic sets. *Ergodic Theory Dynam. Systems*, 25(2):517–551, 2005.
- [16] Michael Field and William Parry. Stable ergodicity of skew extensions by compact Lie groups. *Topology*, 38(1):167–187, 1999.
- [17] Oliver Jenkinson. Smooth cocycle rigidity for expanding maps, and an application to Mostow rigidity. *Math. Proc. Cambridge Philos. Soc.*, 132(3):439–452, 2002.
- [18] Carlangelo Liverani. On contact Anosov flows. *Ann. of Math. (2)*, 159(3):1275–1312, 2004.
- [19] André Martinez. *An introduction to semiclassical and microlocal analysis*. Universitext. Springer-Verlag, New York, 2002.
- [20] Yushi Nakano, Masato Tsujii, and Jens Wittsten. The partially captivity condition for $u(1)$ extensions of expanding maps on the circle. *Preprint*, 2015.
- [21] W. Parry and M. Pollicott. Stability of mixing for toral extensions of hyperbolic systems. *Tr. Mat. Inst. Steklova*, 216(Din. Sist. i Smezhnye Vopr.):354–363, 1997.
- [22] Mark Pollicott. A complex Ruelle-Perron-Frobenius theorem and two counterexamples. *Ergodic Theory Dynam. Systems*, 4(1):135–146, 1984.

- [23] Mark Pollicott. On the mixing of Axiom A attracting flows and a conjecture of Ruelle. *Ergodic Theory Dynam. Systems*, 19(2):535–548, 1999.
- [24] David Ruelle. *Thermodynamic formalism*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2004. The mathematical structures of equilibrium statistical mechanics.
- [25] Michael Ruzhansky and Ville Turunen. *Pseudo-differential operators and symmetries*, volume 2 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser Verlag, Basel, 2010. Background analysis and advanced topics.
- [26] Michael E. Taylor. *Partial differential equations I. Basic theory*, volume 115 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2011.
- [27] Michael E. Taylor. *Partial differential equations II. Qualitative studies of linear equations*, volume 116 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2011.
- [28] Masato Tsujii. Decay of correlations in suspension semi-flows of angle-multiplying maps. *Ergodic Theory Dynam. Systems*, 28(1):291–317, 2008.
- [29] Maciej Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

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